

A Dynamic Theory of the Balance Sheet

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Abstract

We analyze dynamic corporate financing under moral hazard. In the optimal contract, financiers provide funds in exchange for bonds and outside equity. The entrepreneur is incentivized by a fraction of the equity. Retained earnings are the state variable governing the payoff and liquidation decisions. When they reach a prespecified threshold dividends are paid, when they fall to zero the firm is liquidated. We characterize the continuous time dynamics of stock and bond prices. The volatility of stock prices is stochastic, and decreases as the value of the firm increases. Default risk also decreases as the value of the firm increases, but it never goes to zero, and is factored in the bond price. Small firms have high default risk, but large upside potential and thus high market to book ratio. Large firms have high liquidity and low market to book ratios. Finally, leverage decreases as a function of the value of the firm.

1 Introduction

While the theoretical corporate finance literature has generated rich qualitative insights, it has focused, in large part, on one or two-period models.¹ Real world corporate finance problems, and data, are inherently dynamic, however. Furthermore, asset pricing models focus on the dynamics of the value of the claims issued by corporations. The goal of this paper is to analyze corporate finance, under asymmetric information, in a dynamic context. Thus, we attempt to bridge the gap between corporate finance theory on the one hand, and financial data and pricing models on the other hand. To achieve this goal, we extend a standard one period principal agent model of corporate finance to the dynamic case.

In our model, the entrepreneur has discovered an investment project, which only her can manage successfully. She has limited wealth and limited liability. She contacts a "financier", who can be a bank or represent a group of dispersed investors, in order to raise cash for funding the necessary initial investment. Both the entrepreneur and the financier are risk neutral. The latter discounts the future at a lower rate than the former, i.e, the financier is less impatient than the entrepreneur. After the initial investment, the project yields random, positive or negative, observable and contractible cash flows each period. The net present value of the project is positive if the entrepreneur exerts effort at each period. Managerial effort is not observable, however. Thus, there is a moral hazard problem²: the entrepreneur is the agent and the financier the principal. The financial contract must incentivize the entrepreneur to exert effort. We analyze the optimal contract, specifying the payments to the entrepreneur and the financier as well as the decision to liquidate the firm or to continue operating it, as a function of the observable variables.

Our model can be viewed as a generalization of the analysis of Innes (1990) and Holmstrom and Tirole (1997) to the dynamic case. As in Spear and Srivastava (1987), Thomas and Worall (1994) and Phelan and Townsend (1991), we use the continuation payoff of the entrepreneur as the state variable, on which the optimal contract is contingent. The intuition is that, at each point in time, the (unobservable) action of the entrepreneur reflects what she expects to earn in the future. Thus, the optimal contract, regulating the incentives of the manager, is also a function of her expected gains in all the continuation subgames. Also, because we consider an

¹Notable exceptions discussed below include the dynamic analyses of Gromb, 1999, DeMarzo and Fishman, 2002, Albuquerque and Hopenhayn, 2004, Clementi and Hopenhayn, 2002, Quadrini, 2003, Leland, 1994, Leland and Toft, 1996, Anderson and Sundaresan, 1996, and Acharya et al, 2002.

²Our model can also be reinterpreted as a model without effort decision, but where cash flows are privately observable by the manager, as in Diamond (1984), Bolton and Scharfstein (1990) and Gromb (1999). A dynamic version of such a model is studied by DeMarzo and Fishman (2003).

infinite horizon problem, the optimal contract is stationary (rather than time dependent). This simplifies the analysis.

We solve for the optimal contract by the use of dynamic programming techniques: We write the social value of the firm as the solution of a Bellman equation. The command variables are the functions mapping the state variable into the payoffs to the entrepreneur and the financier, and the liquidation decision. We show that there exists a unique solution to this problem and characterize this solution in a discrete time framework. Then we take the continuous time limit.

The optimal contract pins down the dynamics of the continuation payoff of the manager. It evolves stochastically with the cash flows generated by the firm. After positive profits, it increases, while after negative cash flows it decreases. After a series of bad draws, the continuation payoff can reach zero. In that case, the firm is liquidated, i.e. zero is an absorbing barrier. On the other hand, after a sufficiently large number of good draws the continuation payoff reaches its upper bound, which is a reflecting barrier. As long as the state variable remains strictly between these two bounds, the entire cash flow of the firm is collected by the financier. When the state variable reaches the reflecting barrier, the entrepreneur receives a positive payment. The promise of this payment is what drives the incentives of the entrepreneur to exert effort, to generate profits and thus increase the state variable.

The optimal contract can be implemented in a simple and realistic way. The firm issues debt and equity. Debt pays a constant coupon at each period until the firm is liquidated. Equity receives dividends when the state variable reaches the reflecting barrier. Retained earnings, equal to the operational cash flows, minus debt coupons and the dividends paid to shareholders, are deposited on a current account at the bank. The bank commits to remunerate these deposits at a rate that is higher than the market rate, and equal to the manager's discount rate. In exchange for this commitment to remunerate the current account of the firm above market rate, the bank receives an initial payment. At any point in time, the expected continuation pay-off of the entrepreneur remains strictly proportional to the amount of cash reserves on the firm on its bank account, which can also be interpreted as the book value of the firm, and can be taken as an alternative state variable. Our model has the feature that the market value of the firm's equity is always greater than its book value, with equality only when the firm is liquidated. Dividends are paid only when cumulated retained earnings, or reserves, reach a threshold. The firm is liquidated when its current account falls to zero. At this point the value of the firm and the (book or market) value of the equity are also equal to zero.

In exchange for the funding it provides, the financier receives the debt and a fraction of the equity (outside equity). The entrepreneur receives the other fraction (inside equity) The

manager's remuneration, which incentivizes her to exert effort, only consists of her share of the dividends. The greater the magnitude of the moral hazard problem, the greater the need to provide incentives to the manager, the greater the fraction of the equity allocated to the manager. This, in turn, reduces the pledgeable income. As in Holmstrom and Tirole (1997), when the moral hazard problem is too severe and the needed funding too large, the pledgeable income is lower than the need for outside funds, and the firm cannot obtain enough financing.

The characterization of debt and equity as optimal contracts in our dynamic model differs from that arising in one period security design models. In the latter, different securities correspond to different functions (concave for debt and convex for equity) of the (single and final) cash flow. In our model different securities correspond to a different time series of payoffs. Debt pays out a stream of constant coupons, until the firm is liquidated. Equity pays off less frequently and randomly, when the stochastic retained earnings reach the prespecified threshold.

In the optimal contract outside financiers hold bonds and stocks. We analyze the market and book values of these claims. The continuous time limit of our analysis is sufficiently tractable that we can characterize the dynamics of these values. This yields a rich set of empirical implications. When the firm generates a series of positive cash flows, its size, measured by its total balance sheet, increases, and both the market value of debt and that of equity rise. Simultaneously, the firm becomes more liquid, as the ratio of current account to debt increases, and the market to book ratio declines. While positive cash flows reduce the probability of liquidation, default risk is never eliminated.³ Even when the firm fares quite well, there remains a risk that it will decline in the future, and eventually be liquidated. Hence, the bond price always include a default risk premium. Furthermore, leverage, measured as the ratio of the market value of debt to the market value of equity, decreases as a function of the size of the firm. Finally, while the volatility of operational cash flows is constant in our model, the volatility of the stock price is stochastic, and larger for smaller firms.

To give a flavor of the set of implications generated by our theoretical analysis, it can be illustrative to focus on two polar cases which can arise on the equilibrium path: On the one hand, small firms are illiquid, with a high default probability, but a large upside potential. Correspondingly, they have a high market to book ratio. On the other hand, large firms are liquid, they have a low default risk and limited upside potential, and therefore low market to book ratio.

³This contrasts with Clementi and Hopenhayn (2002). In their analysis, after a series of good draws, the firm reaches an absorbing state, where the financier no longer receives any payment, financial markets imperfections cease to matter, and default risk vanishes.

Recent empirical studies document that firms' histories affect their capital structure. For example, Welch (2003) finds that leverage ratios are related to past stock returns. Kayhan and Titman (2004) find that this relation is strong and non linear, and that firms experiencing changes in their stock price do not tend to rebalance their leverage ratios. From the perspective of a static theory of the financial structure of corporations, such as, e.g., the tradeoff theory, this could be interpreted as deviations from optimal capital structures. In contrast, our dynamic theory of optimal financial structure implies these stylized facts.

The present paper is in line with the burgeoning recent literature on dynamic optimal contracting and corporate finance under information asymmetries (see, e.g., Gromb, 1999, Thomas and Worall, 1994, DeMarzo and Fishman, 2002, Albuquerque and Hopenhayn, 2004, Clementi and Hopenhayn, 2002, Quadrini, 2003). To the best of our knowledge, the detailed empirical implications generated by our model for the continuous time dynamics of market to book, leverage, and liquidity ratios, as well as size and default risk are new to the literature. Furthermore, our focus and assumptions differ from those of the above mentioned papers.

Bolton and Scharfstein (1990), Gromb (1999), De Marzo and Fishman (2002) and Clementi and Hopenhayn (2002) analyze the case where, after the cash-flow has been generated, the entrepreneur privately observes it and can divert it instead of paying the financiers. In the same spirit, Thomas and Worall (1994) and Albuquerque and Hopenhayn (2004) assume that, after cash flows have been generated, the entrepreneur can collect them and default on her obligations to the lenders. In contrast, as Innes (1990) and Holmstrom and Tirole (1997), we consider the case where cash flows are observable and contractible, but before they have been generated, the entrepreneur can exert unobservable effort to stochastically improve them. Thus, we focus on an ex-ante moral hazard problem (arising before the cash flow is generated), while the above mentioned papers focus on an ex-post moral hazard problem (arising after cash flows have been generated).⁴

The optimal dynamic financial contracts we obtain differ from those arising in the literature. In DeMarzo and Fishman (2002), the firm is financed with debt and a credit line, while in Albuquerque and Hopenhayn (2004) and in Clementi and Hopenhayn (2002), the firm is financed with short- and long-term debt. Our optimal contract could not be implemented with such financial structures. It involves outside equity, and thus we can analyze the dynamics of the market value of bonds and stocks. On the other hand, important issues, such as renegotiation

⁴Similarly, Quadrini (2003) assumes the entrepreneur can divert some of the resources available to the firm. Another difference between our analysis and that of Clementi & Hopenhayn (2002) is that they assume the financier and the entrepreneur have the same discount rate, while we assume the entrepreneur is more impatient than the financier.

proofness (addressed by Dewatripont et al. 2004, Quadrini, 2003, and DeMarzo and Fishman, 2002), or dynamic investment (addressed by Clementi and Hopenhayn, 2002), are beyond the scope of the present paper.

Our analysis of the dynamics of bond prices in presence of imperfections complements the literature extending the bond pricing model of Merton (1974) to the cases where there are bankruptcy costs (Leland, 1994, Leland and Toft, 1996) or where debtors behave strategically (see, e.g., Anderson and Sundaresan, 1996, Acharya et al, 2002). Our paper differs from these because the financial claims we study arise endogenously as optimal contracts. This enables us to endogenize certain features of their dynamics. For example, that liquidation occurs when the stock price reaches zero is an assumption in Leland (1994), it is a result in our analysis.⁵

The next section presents our model. Section 3 shows existence and uniqueness of the optimal contract in discrete time. Section 4 characterizes this optimal contract. Section 5 presents the continuous time limit of our model. Section 6 shows how the optimal contract can be implemented by issuing debt and outside equity along with a current account management agreement with the bank. Section 7 outlines the empirical implications of our theoretical analysis for the dynamics of securities prices and of the financial structure of corporations. The appendix collects some technical results, useful to establish our propositions.

2 The Model

Consider an entrepreneur operating a project in discrete time over an infinite horizon. This project needs an initial investment I and at each period, it can be continued or liquidated. The liquidation value of the project is 0. At any date $t = 0, 1, 2, \dots$ prior to liquidation, the entrepreneur can decide to exert effort, $e = 1$, in which case the cash-flow is R_1 with probability p and R_0 with probability $1 - p$. If the entrepreneur does not exert effort, $e = 0$, she gets a private benefit B , but the probability of R_1 is only $p - \Delta p < p$. We denote:

$$\begin{aligned}\bar{R} &= pR_1 + (1 - p)R_0, \\ \underline{R} &= (p - \Delta p)R_1 + (1 - p + \Delta p)R_0.\end{aligned}$$

There is no depreciation of capital nor any new investment: the size of physical capital is thus fixed. We focus on the financing of operating expenses: Returns can be understood as cash-flows minus input costs. Thus, after a success, we get a positive return, $R_1 > 0$, while after a failure,

⁵Without information asymmetry, it would be conceivable to write a contract such that, for some paths, the stock price would go to zero, while the firm would be continued, and all its cash flows would accrue to the external financier. With moral hazard this is not feasible. Indeed, the entrepreneur must be promised dividends to incentivize her to exert efforts. Without such efforts it is suboptimal to continue the firm.

we get a negative return, $R_0 < 0$. In the sequel, subscripts 0 or 1 refer to variables contingent on failure (low return R_0) or success (high return R_1). If the entrepreneur exerts effort, expected cash flows at each period are positive:

$$\bar{R} > 0,$$

while if the entrepreneur does not exert effort, the project is not profitable:

$$\underline{R} + B < 0.$$

We also assume that:

$$\frac{pB}{\Delta p} > \bar{R},$$

otherwise the first-best could be achieved in the one-period contract while preserving the lender's participation constraint, by giving the entrepreneur a transfer $B/\Delta p$ in case of success and 0 in case of failure. Both the entrepreneur and the financier are risk-neutral with discount rates r for the financier and $\rho > r$ for the manager. We shall use the following notation:

- w is the state variable representing the expected discounted utility of the manager.
- In any period prior to liquidation, a long-term contract specifies a continuation probability x , a transfer l to the agent in case of liquidation and, conditional on the project not being liquidated, an effort level e , contingent transfers to the manager, u_0 and u_1 , and contingent continuation values for the manager, w_0 and w_1 , depending on the return of the project within that period.

Note that, unlike in the continuous time model of Sannikov (2004), we allow for contingent transfers within the period.

3 The Optimal Contract

We focus on contracts in which the entrepreneur exerts effort in all periods, conditional on the project being continued. We will verify ex-post that this is indeed optimal. The expected discounted revenue of the lender satisfies the following Bellman equation:

$$F(w) = \max \left\{ x \left[\bar{R} - pu_1 - (1-p)u_0 + \frac{pF(w_1) + (1-p)F(w_0)}{1+r} \right] - (1-x)l \right\} \quad (1)$$

for each $w \geq 0$, subject to the constraints (2)–(5) below. First, the contract must satisfy a consistency condition:

$$w = (1-x)l + x \left\{ pu_1 + (1-p)u_0 + \frac{[pw_1 + (1-p)w_0]}{1+\rho} \right\}. \quad (2)$$

Second, the incentive compatibility condition must hold:

$$u_1 + \frac{w_1}{1 + \rho} - u_0 - \frac{w_0}{1 + \rho} \geq \frac{B}{\Delta p}. \quad (3)$$

Finally, the following limited liability conditions are required:

$$\begin{aligned} l &\geq 0, \\ u_0 &\geq 0, \\ u_1 &\geq 0, \\ w_0 &\geq 0, \\ w_1 &\geq 0, \end{aligned} \quad (4)$$

as well as the feasibility condition:

$$0 \leq x \leq 1. \quad (5)$$

It is convenient to introduce an auxiliary function $V(w) = w + F(w)$. This function satisfies the following, simpler, Bellman equation:

$$V(w) = \max \left\{ x \left\{ \bar{R} + \frac{pV(w_1) + (1-p)V(w_0)}{1+r} - \frac{(\rho-r)[pw_1 + (1-p)w_0]}{(1+r)(1+\rho)} \right\} \right\} \quad (6)$$

for each $w \geq 0$, subject to the constraints (2)–(5). It should be noted that, owing to the difference between the manager's and the lender's discount rates, $V(w)$ cannot be interpreted as the value of the firm, unlike in Clementi and Hopenhayn (2002). Since l , u_0 and u_1 do not appear in (6), it will be useful to eliminate them from the constraints as well. Given a triple $(x, w_0, w_1) \in [0, 1] \times \mathbb{R}_+^2$, one can find a triple (l, u_0, u_1) satisfying (2)–(5) if and only if the set of $(u_0, u_1) \in \mathbb{R}_+^2$ such that:

$$\begin{aligned} u_1 - u_0 &\geq \frac{B}{\Delta p} - \frac{w_1 - w_0}{1 + \rho}, \\ w - \frac{x[pw_1 + (1-p)w_0]}{1 + \rho} &\geq x[pw_1 + (1-p)u_0] \end{aligned}$$

is non-empty. This is satisfied if and only if:

$$w \geq x \left(\frac{w_0}{1 + \rho} + \frac{pB}{\Delta p} \right), \quad (7)$$

$$w \geq \frac{x[pw_1 + (1-p)w_0]}{1 + \rho}, \quad (8)$$

Thus the constraints (2)–(5) are equivalent to constraints (7), (8) together with:

$$(x, w_0, w_1) \in [0, 1] \times \mathbb{R}_+^2. \quad (9)$$

It follows immediately from (4) and (7) that for $w < pB/\Delta p$, liquidation must occur with positive probability, and that the probability of liquidation tends to 1 as w goes to zero. One has the following results, proved in the appendix.

Proposition 1. *There exists a unique continuous and bounded solution V to (6) subject to (7)–(9). V is increasing, concave sl , $V(0) = 0sl$, and there exists $w^* > 0$ such that V is strictly increasing over $[0, w^*)sl$ and constant over $[w^*, \infty)$. Moreover sl , it is never optimal to set $l > 0$.*

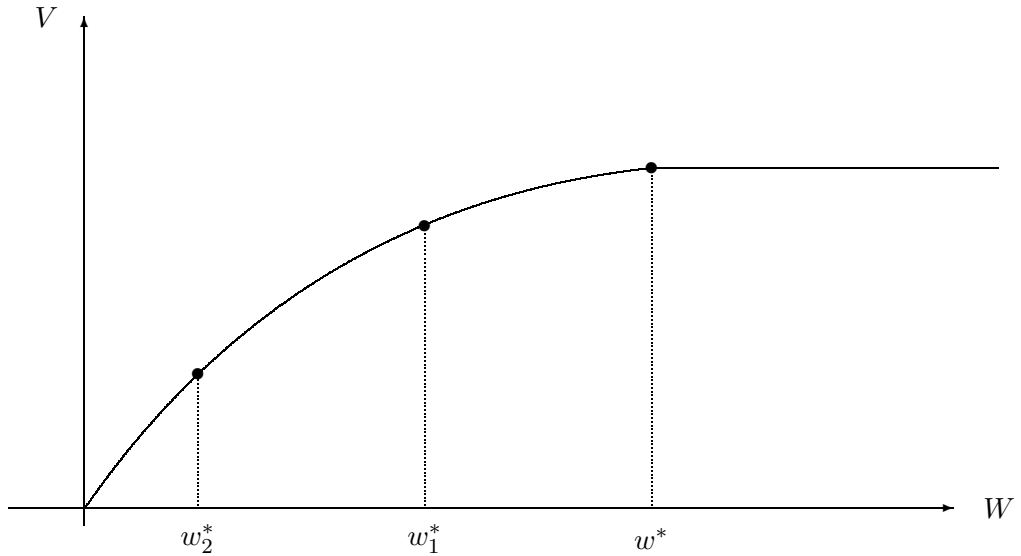


Figure 1: The value function.

A key consequence of Proposition 1 is as follows. Let \hat{w} be the smallest point at which the mapping $w \mapsto V(w) - (\rho - r)w / (1 + \rho)$ reaches its maximum, and define $w^* = \hat{w} / (1 + \rho) + pB / \Delta p$. Then, if $w \geq w^*$, it is optimal⁶ to never liquidate the project, $x = 1$, to make contingent transfers $u_0 = w - w^*$ and $u_1 = w - w^* + B / \Delta p$, and to set continuation values $w_0 = w_1 = \hat{w}$. In particular, incentives are then only provided through contingent transfers within the period, and not through continuation values. The following result relates the values of \hat{w} and w^* to the fundamentals of the problem.

Proposition 2. *For any $\rho > r$,*

$$\hat{w} < w^* < \frac{(1+r)\bar{R}}{\rho-r} + \frac{pB}{\Delta p}sl.$$

⁶Note that if $w > w^*$, there are other pairs (u_0, u_1) that implement $x = 1$ and $w_0 = w_1 = \hat{w}$. For $w = w^*$, this is the only optimal contract.

Since $\hat{w} < w^*$, and unlike in Clementi and Hopenhayn (2002), or in the continuous time model of Sannikov (2004), w^* is not an absorbing point for the manager's expected discounted utility. In particular, there is no region of the state space in which the financial constraints cease to bind. The upper bound on w^* will be useful when we consider the continuous time limit of the discrete time model.

Before we proceed with the main results, it will be useful to dispose of two special cases of the analysis. The following result delineates the circumstances under which $\hat{w} = 0$.

Proposition 3. *Suppose that:*

$$\bar{R} \leq \frac{(\rho - r)pB}{(1 + \rho)\Delta p}. \quad (10)$$

Then $\hat{w} = 0$ and:

$$V(w) = \begin{cases} [\bar{R}/(pB/\Delta p)]w & \text{if } w \leq w^*, \\ \bar{R} & \text{if } w \geq w^*. \end{cases} \quad (11)$$

Conversely, if $\hat{w} = 0$, then (10) holds.

Under (10), the optimal contract is characterized by two regions. If $w \geq w^*$, it is optimal to operate the project with certainty, while if $w < w^*$, the project is operated with probability w/w^* and liquidated with probability $1 - w/w^*$. In any case, it is optimal to set $w_0 = w_1 = 0$, and, conditional on continuation, to make contingent transfers $u_0 = \max\{w - w^*, 0\}$ and $u_1 = \max\{w - w^*, 0\} + B/\Delta p$. This case is degenerate in the sense that the investment capacity of the firm is 0. Indeed, since $\bar{R} < pB/\Delta p$, the slope of V is strictly less than 1, which implies that $F(w) = V(w) - w < 0$ for each $w > 0$.

If (10) does not hold, then $\hat{w} > 0$. It is easy to see that one must then have $\hat{w} \geq pB/\Delta p$, since V is necessarily linear over $[0, pB/\Delta p]$, and \hat{w} is defined as the smallest point at which the mapping $w \mapsto V(w) - (\rho - r)w/(1 + \rho)$ reaches its maximum. The following result delineates the circumstances under which $\hat{w} = pB/\Delta p$.

[TO BE COMPLETED]

4 The Different Regimes

In this section, we describe in more detail the different regimes that characterize the optimal contract in discrete time in the more interesting case where $\hat{w} > \frac{pB}{\Delta p}$. The next section will study the continuous time limit of this optimal contract.

Recall that we denote $\hat{V}(w) = V(w) - \frac{\rho-r}{1+\rho}w$ and that the Bellman equation writes:

$$V(w) = \max x \left[\bar{R} + \frac{p\hat{V}(w_1) + (1-p)\hat{V}(w_0)}{1+r} \right], \quad (12)$$

under the constraints:

$$0 \leq x \leq 1 \quad w_0 \geq 0 \quad w_1 \geq 0 \quad (\text{never binding})$$

$$\begin{aligned} \left(\frac{w_0}{1+\rho} + \frac{pB}{\Delta p} \right) x &\leq w \quad (\text{multiplier: } (1-p)\lambda_0) \\ \left(\frac{pw_1 + (1-p)w_0}{1+\rho} \right) x &\leq w \quad (\text{multiplier: } \lambda_1). \end{aligned}$$

The Lagrangian is:

$$\begin{aligned} L(w) &= x \left[\bar{R} + \frac{p\hat{V}(w_1) + (1-p)\hat{V}(w_0)}{1+r} - \lambda_1 \left(\frac{pw_1 + (1-p)w_0}{1+\rho} \right) - \lambda_0(1-p) \left(\frac{w_0}{1+\rho} + \frac{pB}{\Delta p} \right) \right] \\ &\quad + w[\lambda_1 + \lambda_0(1-p)]. \end{aligned}$$

The first order conditions can be written:

$$\begin{aligned} \frac{\partial L}{\partial w_1} &= \frac{p}{1+r} \left[\hat{V}'(w_1) - \lambda_1 \frac{1+r}{1+\rho} \right] = 0 \\ \frac{\partial L}{\partial w_0} &= \frac{1-p}{1+r} \left[\hat{V}'(w_0) - (\lambda_0 + \lambda_1) \frac{1+r}{1+\rho} \right] \leq 0 \quad (= \text{ if } w_0 > 0) \\ \frac{\partial L}{\partial x} &= \left[\bar{R} + \frac{p\hat{V}(w_1) + (1-p)\hat{V}(w_0)}{1+r} - \lambda_1 \left(\frac{pw_1 + (1-p)w_0}{1+\rho} \right) - \lambda_0(1-p) \left(\frac{w_0}{1+\rho} + \frac{pB}{\Delta p} \right) \right] \end{aligned}$$

We now characterize the different regimes, starting by the largest values of w .

1st Regime: It is characterized by the absence of frictions. The incentive compatibility and limited liability constraints do not bind: $\lambda_0 = \lambda_1 = 0$. The continuation probability is maximum: $x = 1$; The continuation values for the manager are optimal: $w = w_0 = \hat{w}$ where $\hat{w} = \arg \max \hat{V}$.

$V(w) = \max V$ in this regime.

This is feasible iff $w \geq w^* \equiv \frac{\hat{w}}{1+\rho} + \frac{pB}{\Delta p}$.

Payments to the agent are: $u_0 = w - w^*$, $u_1 = w - w^* + \frac{B}{\Delta p}$.

Notice that this regime is never attained after date 0, since $\hat{w} < w^*$.

2nd Regime: In this regime, the agent receives a payment, but only in case of high return. The limited liability constraint of the manager binds, but only in the low return state. The continuation probability is maximal. This regime is thus characterized by:

$$\begin{aligned} \lambda_1 &= 0, \lambda_0 > 0, x = 1; \\ w_1 &= \hat{w}, w_0 = (1+\rho) \left(w - \frac{pB}{\Delta p} \right). \end{aligned}$$

$$u_0 = 0, u_1 = w - w^* + \frac{B}{\Delta p}.$$

This is feasible if $w \geq w_1^* = w^* - \frac{B}{\Delta p}$ (this guarantees that $u_1 > 0$) and $w < w^*$ (this guarantees that $\lambda_0 > 0$). In case of high return, the firm stays in this regime and entrepreneur receives a payment. In case of low return, not only the agent receives no payment but also the firm moves to the lower, third regime that we now describe.

3rd Regime: In this regime the agent receives no payment, even in case of high returns and the limited liability constraint binds in both states. The continuation probability is one. Thus we have:

$$\lambda_0 > 0, \lambda_1 > 0, x = 1;$$

The continuation pay off increases on average at rate ρ , but there is a positive covariance with the returns

$$w_0 = (1 + \rho) \left(w - \frac{pB}{\Delta p} \right) \quad (13)$$

$$w_1 = (1 + \rho) \left(w + \frac{(1-p)B}{\Delta p} \right). \quad (14)$$

In this regime incentives are only provided by the continuation pay-offs. As is Sannikov (2004) there is a minimum volatility of w_t that is needed to provide incentives for effort. Indeed the above equations can be rewritten as:

$$w_{t+1} - w_t = \rho w_t + (1 + \rho) \frac{B}{\Delta p} \epsilon_t$$

where ϵ_t equals $(1 - p)$ when returns are high and $-p$ when they are low.

This is feasible if

$$w \geq \frac{pB}{\Delta p} \quad \text{and} \quad \frac{\partial L}{\partial x} \geq 0.$$

Now

$$\frac{\partial L}{\partial x} \equiv \varphi(w) = \bar{R} + \frac{p\hat{V}(w_1(w)) + (1-p)\hat{V}(w_0(w))}{1+r} - w[\lambda_1(w) + (1-p)\lambda_0(w)],$$

where $w_1(w)$ and $w_0(w)$ are given by (13) and (14) and

$$\lambda_1(w) = \frac{1+\rho}{1+r} \hat{V}'[w_1(w)], \quad (15)$$

$$\lambda_0(w) = \frac{1+\rho}{1+r} \{\hat{V}'[w_0(w)] - \hat{V}'[w_1(w)]\} \quad (16)$$

This function of w is defined for $w \geq \frac{pB}{\Delta p}$ and we have:

$$\begin{aligned}\varphi'(w) &= \frac{1+\rho}{1+r} [p\hat{V}'(w_1) + (1-p)\hat{V}'(w_0)] - [\lambda_1(W) + (1-p)\lambda_0(W)] \\ &\quad - w[\lambda_1'(w) + (1-p)\lambda_0'(W)].\end{aligned}$$

After simplification:

$$\varphi'(w) = -w[\lambda_1'(w) + (1-p)\lambda_0'(W)] > 0,$$

by (15) and concavity of \hat{V} .

Thus either $\varphi' \left(\frac{pB}{\Delta p} \right) > 0$ (in which case $w_2^* \equiv \frac{pB}{\Delta p}$ is the lower end of the third regime) or $\varphi' \left(\frac{pB}{\Delta p} \right) \leq 0$, in which case $w_2^* = \varphi^{-1}(0)$.

4th Regime: The last regime corresponds to financial distress: not only the agent receives no payment, but the probability of continuation is less than one. The threat of liquidation is used as an additional instrument to provide incentives for effort. Therefore $\lambda_0 > 0$, $\lambda_1 > 0$, $x < 1$; then (13) and (14) hold true with $\frac{w}{x}$ replacing w , which means that

- either $w_2^* = \varphi^{-1}(0) > \frac{pB}{\Delta p}$: then $\frac{\partial L}{\partial x} = \varphi \left(\frac{w}{x} \right) = 0$ and $x = \frac{w}{w_2^*}$,
- or $w_2^* = \frac{pB}{\Delta p}$ then $\lambda_0 \geq \frac{1+\rho}{1+r} \hat{V}'(0)$ λ_0 is then determined by the other condition $\frac{\partial L}{\partial x} = 0$.
Also in this case, we have $x = \frac{W}{W_2^*}$.

Thus in all cases:

$$w_0 = (1+\rho) \left[w_2^* - \frac{pB}{\Delta p} \right],$$

and

$$w_1 = (1+\rho) \left[W_2^* + \frac{(1-p)B}{\Delta p} \right]$$

are constant! This implies that V is linear over this last regime:

$$V(w) = \frac{w}{w_2^*} \left[\bar{R} + \frac{p\hat{V}(w_1) + (1-p)\hat{V}(w_0)}{1+r} \right].$$

5 The Continuous Time Limit

We now introduce a time interval h between periods and look at the limit when $h \rightarrow 0$, in the aim to study the continuous time version of the above model. So we let the parameters of

the model be functions of h such that the mean and variance of returns per unit of time are preserved. Namely we require:

$$\begin{cases} pR_1 + (1-p)R_0 & = \mu h \\ \Delta p(R_1 - R_0) & = \Delta\mu h \\ p(1-p)(R_1 - R_0)^2 & = (p - \Delta p)(1 - p + \Delta p)(R_1 - R_0)^2 = \sigma^2 h. \end{cases}$$

For h small, there is a unique solution to this system, namely:

$$p = \frac{1}{2} \left[1 + \frac{h\Delta\mu}{\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}} \right], \quad (17)$$

$$\Delta p = \frac{h\Delta\mu}{\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}}, \quad (18)$$

$$R_0 = \left(\mu - \frac{1}{2}\Delta\mu\right)h - \frac{1}{2}\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}, \quad (19)$$

$$R_1 = \left(\mu - \frac{1}{2}\Delta\mu\right)h + \frac{1}{2}\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}. \quad (20)$$

In this section, we look at the behavior of the optimal contract when p , Δp , R_0 and R_1 are given by formulas (17) to (20), B , r and ρ are multiplied by h , and h converges to zero. For the moment we assume that the Bellman function V_h converges to a well defined \mathcal{C}^2 function V (we prove it later) and characterize the properties of this function and the associated optimal contract.

5.1 The Limit Regimes

Recall that the optimal contract comprises 4 regimes, associated to the thresholds

$$w = 0, \quad w = w_2^*, \quad w = w_1^*, \quad w = w^*.$$

Proposition 4. *When h converges to zero, w_2^* also converges to zero, while w_1^* , and w^* converge to the same limit, characterized by sl :*

$$V'(w^*) = V''(w^*) = 0. \quad (21)$$

Proposition 4 implies that, in the limit, the second and fourth regime degenerate to a singleton:

- when $w = 0$ (fourth regime) the firm is liquidated with probability 1 (absorbing state),
- when $w = w^*$ (second regime) the agent receives a payment (but only in case of success) and w bounces back to the third regime ($0 < w < w^*$). w^* is thus a reflecting barrier. The dividend process converges to a fraction of the local time associated to the passage of the continuous time process w_t at w^* (see Karatzas and Shreve 1998 for a formal definition)

The two other regimes remain the same:

- For $w > w^*$, the agent receives a lump sum payment $w - w^*$ (plus a flow payment in case of success). This regime is never reached after $t = 0$.
- For $0 < w < w^*$ the agent receives no payment but w increases on average at rate ρ , with the minimum volatility $\frac{B}{\Delta\mu}\sigma$, needed to provide incentives for effort (as in Sannikov, 2004).

5.2 The Bellman equation in continuous time

Given the limit regimes characterized above, we can restrict ourselves to the interval $[0, w^*]$. The boundary conditions for V are: $V(0) = 0$, and $V'(w^*) = V''(w^*) = 0$ (supercontact). The Bellman equation in continuous time is obtained through a Taylor expansion of equation (12), where $x = 1$, and w_0, w_1 are given by equations (13) and (14):

$$V_h(w) = \mu h + \frac{p\hat{V}_h(w_1) + (1-p)\hat{V}_h(w_0)}{1+rh},$$

where

$$w_0 = (1+\rho h) \left(w - \frac{pB}{\Delta p} \right), \quad w_1 = (1+\rho h) \left(w + \frac{(1-p)B}{\Delta p} \right).$$

A second order Taylor expansion of \hat{V}_h around w_1 and w_0 gives:

$$\begin{aligned} \hat{V}_h(w_1) &\sim V_h(w) - (\rho - r)hw + [V'_h(w) - (\rho - r)h][w_1 - w] \\ &\quad + \frac{1}{2}V''_h(w)(w_1 - w)^2 \\ \hat{V}_h(w_0) &\sim V_h(w) - (\rho - r)hw + [V'_h(w) - (\rho - r)h][w_0 - w] \\ &\quad + \frac{1}{2}V''_h(w)(w_0 - w)^2. \end{aligned}$$

Thus

$$\begin{aligned} p\hat{V}_h(w_1) + (1-p)\hat{V}_h(w_0) &\sim V_h(w) - (\rho - r)hw + [V'_h(w) - (\rho - r)h] \\ &\quad [(pw_1 + (1-p)w_0) - w] + \frac{1}{2}V''_h(w)[p(w_1 - w)^2 + (1-p)(w_0 - w)^2]. \end{aligned}$$

Using (17) and eliminating terms in $h\sqrt{h}$ and h^2 we get in the limit:

$$rV(w) = \mu - (\rho - r)w + \rho wV'(w) + \frac{1}{2}k^2\sigma^2V''(w), \quad (22)$$

where $k = \frac{B}{\Delta\mu} \in [0, 1]$ parametrizes the intensity of moral hazard.⁷

Proposition 5. *The optimal contract in continuous time is characterized by the following features:*

⁷ k cannot be greater than one: otherwise effort would not be socially optimal.

- On the interval $]0, w^*[$, w_t increases on average at rate ρ with a volatility σ :

$$dw_t = \rho w_t dt + k \sigma dZ_t, \text{ where } dZ_t \text{ is a standard Brownian motion}$$

- When $w_t = 0$, the firm is liquidated (absorbing state).
- When $w_t = w^*$, the agent receives a payment only in case of high return and w_t bounces back in case of low return.

Formally the payment to the agent is given by the local time associated to (19) at $w_t = w^*$. This local time is denoted $L_t^{w^*}$.

6 Implementation

In the continuous time limit, the optimal contract is thus the following:

- At $t = 0$, w is set at the level w_0 implicitly defined as the largest solution of the equation:

$$V(w_0) - w_0 = I. \tag{23}$$

Such a w_0 is the maximal expected income which can be granted to the agent subject to the principal breaking even.

- For $t > 0$, w evolves according to the stochastic differential equation

$$dw_t = \rho w_t dt + k[dX_t - \mu dt] - kdL_t^{w^*}$$

where dX_t is the cash flow process in the limiting case where h goes to zero: $dX_t = \mu dt + \sigma dZ_t$.

- Whenever w_t hits 0, the firm is liquidated and the continuation pay-offs are zero for both the principal and the agent.
- Whenever w_t hits w^* , the agent receives $kdL_t^{w^*}$, and w_t bounces back.

Interestingly, this optimal contract admits a simple and realistic implementation. This implementation requires two securities (a stock and a bond), a cash management contract signed with a bank, and a liquidation rule. The stock is a claim to dividends. Dividends are paid out at the stochastic dates at which the firm's reserves hit a given threshold. In contrast, the bond

is a claim to a continuous stream of constant coupons. The firm is liquidated when it runs out of cash and thus defaults on the bond.

Our implementation introduces two distinct categories of outsiders. Outside equity and the bond are held by a dispersed investors base who cannot renegotiate, and who make no payments to the firm after their initial investment. In contrast, the bank commits to future payments to the firm for an initial fee. Notice that other implementations of the optimal contract are possible. We focus on this one because it appeals to standard financial instruments and delivers a stylized balance sheet with appealing features.

Before proceeding to the formal exposition of this implementation, we introduce the process $m_t = \frac{w_t}{k}$, where w_t is the continuous time limit of the agent's continuation payoff. We let $m^* = \frac{w^*}{k}$. We also define $L_t^{m^*}$ to be the local time associated to m_t in m^* , and $\tau = \inf \{t/m_t = 0\}$. τ is a stopping time adapted to (\cdot) . The process m_t will turn out to play a key role in the implementation: m_t is actually the cash available to the firm at date t . We shall now define formally the stock, the bond, and the bank account.

6.1 Stock

The stock is a claim to a dividend process. The dividend process is $(1_{\{t < \tau\}} \times L_t^{m^*})$. Let S_t denote the market value of the stock:

$$S_t = E \left[\int_t^\tau e^{-r(s-t)} dL_s^{m^*} \right].$$

Thus, $S_t = S(m_t)$, where S is the solution of the differential equation:

$$\begin{cases} rS(m) &= \rho m S'(m) + \frac{\sigma^2}{2} S''(m) \\ S(0) &= 0, \quad S'(m^*) = 1. \end{cases}$$

S_t obeys

$$dS_t = rS_t dt + \sigma(S_t) dZ_t - dL_t^{m^*}, \quad (24)$$

where $\sigma(S) \equiv S'[m(S)]$, and $m(\cdot)$ is the inverse of S .

6.2 Bond

The bond is a claim to an instantaneous payment equal to $1_{\{t < \tau\}} \times \mu$. Let D_t denote the market value of the bond:

$$D_t = E \left[\int_t^\tau e^{-r(s-t)} \mu dt \right] = \frac{\mu}{r} [1 - E_t(e^{-r(\tau-t)})].$$

$D_t = D(m_t)$, where the function D solves:

$$\begin{cases} rD(m) &= \rho m D'(m) + \frac{\sigma^2}{2} D''(m) + \mu \\ D(0) &= 0 \quad D'(m^*) = 0. \end{cases}$$

D_t obeys $dD_t = rD_t dt + \sigma(D_t) dZ_t - \mu dt$, where $\sigma(D_t) = D'(D^{-1}(D_t))$.

Notice that $V(m) = D(m) - kS(m) + km$, so that $I = D_0 - kS_0$.

6.3 Bank Account

The firm enters into an agreement with a bank whereby the bank remunerates a current account at the instantaneous rate ρ for an initial fee. Operational profits are credited to the account. Operational losses, coupons and dividends are debited from the account. Thus, the current balance at any date is equal to the initial deposit plus retained earnings capitalized at the rate ρ . Note that since $\rho > r$, the banker's commitment to deliver a return ρ has a nonnegative value. The initial fee paid to the bank is equal to the value of this commitment.

The following Proposition shows that these three instruments suffice to implement the optimal contract when combined with a natural liquidation rule: firms are liquidated as soon as they are illiquid.

Proposition 6 *The optimal contract can be implemented as follows.*

- *At $t = 0$, the firm issues the bond and the stock. A fraction k of the stock is granted to the insider.*
- *The proceeds from placing outside equity and the bond are used to sink the initial investment I , and deposit m_0 into the bank account.*
- *As long as the firm is operating, operational earnings net of dividends and coupons are credited to the bank account. The firm is liquidated when there is no cash left at the bank.*

Proof. By construction, the balance of the bank account at date t , b_t , solves

$$\begin{aligned} db_t &= \rho b_t + \mu dt + \sigma dZ_t - \mu dt - dL_t^{m^*} \\ b_0 &= m_0 \end{aligned}$$

Comparing with the dynamics of w_t , it is transparent that $b_t = \frac{w_t}{k} = m_t$. This entails that inside equity replicates the payments to the insider specified by the optimal contract, and that the liquidation rule replicates the termination rule of the optimal contract.

It remains to check that the date-0 operations are balanced. The proceeds from issuing outside equity and the bond are equal to $(1 - k)S_0 + D_0$. We know that $I = D_0 - kS_0$. An amount m_0 is credited to the bank account, and the bank's fee f is paid. We have to check that

$$(1 - k)S_0 + D_0 = D_0 - kS_0 + m_0 + f,$$

or

$$f = S_0 - m_0.$$

By definition,

$$f = E \left[\int_0^\tau (\rho - r)m_t e^{-rt} dt \right],$$

and, from $dm_t = \rho m_t + \sigma dZ_t - dL_t^{m^*}$, we have $m_t = e^{\rho t} \int_t^\tau e^{-\rho s} (dL_s^{m^*} - dZ_s)$. Thus,

$$\begin{aligned} f &= E \left[\int_0^\tau (\rho - r) e^{(\rho - r)t} \int_t^\tau e^{-\rho s} (dL_s^{m^*} - dZ_s) dt \right] \\ &= E \left[\int_0^\tau (e^{-rs} - e^{-\rho s}) (dL_s^{m^*} - dZ_s) \right] = S_0 - M_0 \end{aligned}$$

■

The balance sheet of the firm at date 0 is the following:

<u>Assets</u>	<u>Liabilities</u>
m_0	D_0
$I = D_0 - kS_0$	
kS_0	$S_0 - f$

On the asset side, this balance sheet features illiquid productive assets (I) and the liquidities deposited at the bank (m_0). kS_0 is an intangible asset. It may, indeed, be interpreted as the goodwill of the firm. It is the value of the agent's future rents, and thus measures the expected value created by operations since outside financiers break even. The liability side features debt, as well as inside and outside equity. The initial capital is equal to the value of issued stocks minus the initial expense f . Debt and equity differ not only with respect to the seniority of the claims they generate, as in static models. The dynamics of coupons and dividends are also dramatically different. The instantaneous coupon is nonnegative until liquidation, while the instantaneous dividend process $dL_t^{m^*}$ is pathwise equal to zero, except on a negligible set of dates. Default on debt occurs when the firm is illiquid, and this triggers liquidation. As a result, the firm's liquidities obey a diffusion with a reflecting barrier in m^* , where dividends are paid out, and an absorbing barrier in 0 where liquidation occurs.

7 Implications: Relating the Dynamics of Assets Prices to the Financial Structure

This implementation offers a stylized balance sheet in which all items are endogenous, as well as tractable dynamics for the stock, the bond and the bank account. This gives rise to a number of implications regarding the relationship between the financial structure of a firm and the dynamics of the prices of its securities. Analyzing this relationship in a continuous-time environment in which stocks and bonds arise endogenously is unprecedented, to our knowledge. The following Lemma is instrumental in this analysis:

Lemma 1 *The functions $V(m)$, $D(m)$ and $S(m)$ are nondecreasing and concave over $[0, m^*]$.*

Proof. These functions being nondecreasing stems directly from the fact that τ and L^{m^*} increase stochastically with respect to m_t . V is concave as a limit of concave functions. Let us show that S and D are concave. Recall that these functions are defined as follows:

$$\begin{cases} rS(m) &= \rho m S'(m) + \frac{\sigma^2}{2} S''(m) \\ S(0) &= 0, \quad S'(m^*) = 1. \\ rD(m) &= \rho m D'(m) + \frac{\sigma^2}{2} D''(m) + \mu \\ D(0) &= 0 \quad D'(m^*) = 0. \end{cases}.$$

Let us show that S is concave. Deriving the equation satisfied by S , we get that at any point where $S'' = 0$, $sgS''' = sg[(r - \rho)S'] < 0$. Thus, S'' is nonincreasing at its zeros over $(0, m^*)$ if any. The differential equation shows that $S''(m) = \frac{2}{\sigma^2}(r - \rho)S'(0)m + 0(m)$ and is thus nonpositive in 0^+ , because $S'(0) = 0$ would entail $S = 0$. Thus $S'' < 0$ over $(0, m^*)$ necessarily.

To show that D is concave, let us derive again the equation satisfied by D . We get that at any point where $D'' = 0$, $sgD''' = sg[(r - \rho)D'] < 0$. Thus, D'' is nonincreasing at its zeros over $(0, m^*)$ if any. The differential equation shows that $D''(0) < 0$, thus $D'' < 0$ over $(0, m^*)$ necessarily. ■

The functions S and D are depicted below:

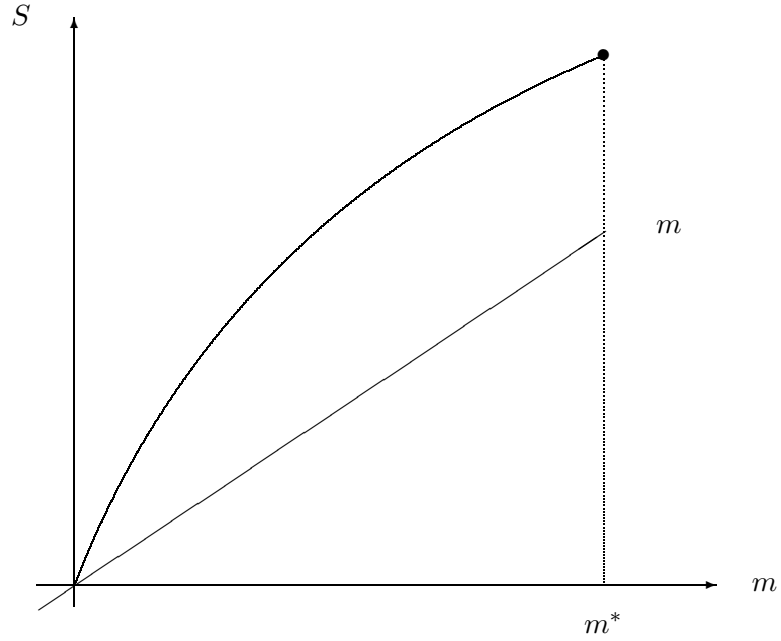


Figure 1: Stock price as a function of m .

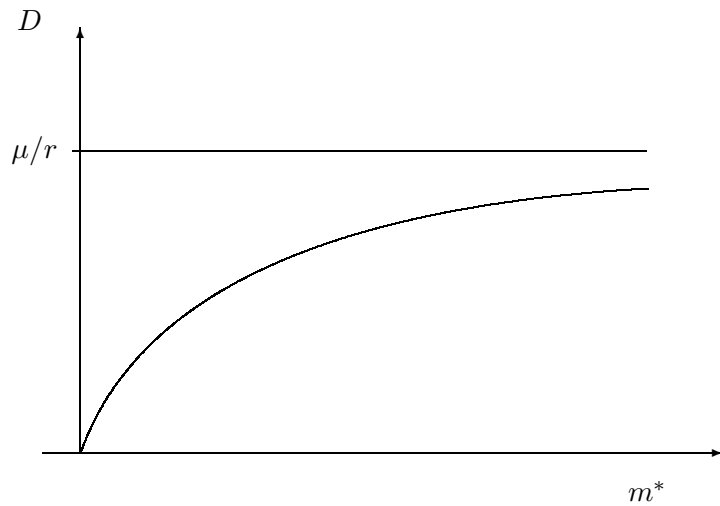


Figure 2: Bond price as a function of m .

Notice that, m^* being a reflecting barrier, the risk premium on D is never zero, even when $m = m^*$.

Lemma 1 has the following dynamic implications:

Proposition 8

1. *Stocks prices are more volatile than earnings. Unlike earnings, stocks prices have a stochastic volatility, which is decreasing with respect to the firm's liquidities m_t .*
2. *The probability of default on the bond is nonnegative over $[0, m^*]$. Unlike earnings, bonds prices have a stochastic volatility, which is decreasing with respect to liquidity m_t .*

3. The leverage expressed in market values, $\frac{D_t}{S_t}$, is a decreasing function of liquidity m_t .
4. The book-to-market ratio $\frac{m_t - kS_0}{S_t}$ increases with respect to liquidity m_t .

Proof.

1., 2., 4. are straightforward consequences of Lemma 1.

To prove 3., let $N(m) = D'(m)S(m) - D(m)S'(m)$ denote the numerator of $(\frac{D}{S})'$. Since

$$\begin{cases} rS(m) &= \rho m S'(m) + \frac{\sigma^2}{2} S''(m) \\ rD(m) &= \rho m D'(m) + \frac{\sigma^2}{2} D''(m) + \mu \end{cases},$$

$$\rho m N = -\frac{\sigma^2}{2} N' - \mu S.$$

Thus, $N = 0 \rightarrow N' < 0$ and N admits at most one zero. Note that

$$\begin{aligned} S''(0) &= 0, \\ S'''(0) &= 2\frac{r-\rho}{\sigma^2} S'(0), \\ D''(0) &= -\frac{2\mu}{\sigma^2}, \\ D'''(0) &= 2\frac{r-\rho}{\sigma^2} D'(0). \end{aligned}$$

Thus,

$$\begin{aligned} D'(m) &= D'(0) - \frac{2\mu}{\sigma^2} m + \frac{r-\rho}{\sigma^2} D'(0) m^2 + 0(m^2), \\ S(m) &= S'(0) m + 0(m^2), \\ D(m) &= D'(0) m - \frac{\mu}{\sigma^2} m^2 + 0(m^2), \\ S'(m) &= S'(0) + \frac{r-\rho}{\sigma^2} S'(0) m^2 + 0(m^2) \end{aligned}$$

As a result, $N(m) = -\frac{\mu}{\sigma^2} S'(0) m^2 + 0(m^2)$ is nonpositive in 0^+ . ■

Our model generates stochastic volatilities for stocks and corporate bonds prices even though (i) earnings have a constant volatility, (ii) the stochastic discount factor is deterministic, the risk-free interest rate is constant, (iii) and the capital structure and the technology are fixed. Stochastic volatility is the mere consequence of a standard moral hazard problem. Stocks prices exhibit excess volatility. The volatility of the stock price is decreasing with the stock price and the liquidity of the firm m_t . It is equal to the volatility of earnings at $m_t = m^*$, when the dividends are paid out.

Because of the reflecting barrier in m^* , the probability of default on bonds is bounded away from 0 regardless of the history of the firm. The volatility of bonds prices tends to 0 when this barrier is hit.

Stock returns are always more volatile than bond returns: The decreasing leverage means that $\frac{S'}{S} > \frac{D'}{D}$.

Finally, the book value of equity B_t is equal to cash plus productive assets at their purchase value net of historical value of debt:

$$B_t = I + m_t - D_0 = m_t - kS_0.$$

The book-to-market ratio is increasing with respect to liquidity. Thus, the evolution of the book-to-market ratio and the stock return would be positively correlated in any panel data set generated by our model.

To sum up, our model generates two types of firms. Firms that have been lucky have a value profile. They have a large and liquid balance sheet, a high-book-to-market ratio, a low probability of failure. They pay dividends frequently and the prices of their securities have a low volatility. Firms that have been unlucky have a growth profile. They are illiquid, with a low book-to-market equity, have not paid many dividends. They have a high probability of default and their stocks and bonds have volatile prices.

The following Proposition derives some comparative statics properties of the initial balance sheet:

Proposition 7 *The maximal investment capacity the firm can afford, I^* , is increasing with respect to μ , decreasing with respect to σ and k .*

For $I \leq I^$, the initial amount of liquid assets m_0 , and the liquidity ratio $\frac{m_0}{I}$ are increasing with respect to μ , decreasing with respect to σ and k . Inside equity kS_0 increases with respect to k .*

Proof. By definition, $I^* = \max_m F(m) = D(m) - kS(m)$, where F is the solution of

$$\begin{cases} rF(m) &= \rho mF'(m) + \frac{\sigma^2}{2}F''(m) + \mu \\ F(0) &= 0, F'(m^*) = -k, F''(m^*) = 0 \end{cases}.$$

$F(m) = V(m) - km$ and is thus concave. We study the variations of $F(m, \mu, \sigma^2, k)$ with respect to μ, σ^2 , and k .

Let $H = \frac{\partial F}{\partial \mu}$. H solves

$$\begin{cases} rH(m) &= \rho mH'(m) + \frac{\sigma^2}{2}H''(m) + 1 \\ H(0) &= 0 \quad H'(m^*) = 0. \end{cases}.$$

Thus, $H(m) = E\left(\int_0^\tau e^{-rt} dt\right) = \frac{D_0}{\mu} > 0$. I^* increases with respect to μ .

Let $G = \frac{\partial F}{\partial k}$. G solves

$$\begin{cases} rG(m) &= \rho mG'(m) + \frac{\sigma^2}{2}G''(m) \\ H(0) &= 0 \quad H'(m^*) = -1. \end{cases}.$$

Thus, $H(m) = -S_0 < 0$. Thus I^* decreases with respect to k .

Let $J = \frac{\partial F}{\partial \sigma^2}$. J solves

$$\begin{cases} rJ(m) &= \rho m J'(m) + \frac{\sigma^2}{2} J''(m) + \frac{1}{2} F''(m) \\ J(0) &= 0 \quad J'(m^*) = 0. \end{cases}$$

Thus, $H(m) = E \left(\int_0^\tau \frac{F''(t)}{2} e^{-rt} dt \right) < 0$. Thus I^* decreases with respect to σ^2 .

Now, since $m_0 = F^{-1}(I)$ and F is concave, the comparative statics of m_0 and $\frac{m_0}{I}$ are similar to the comparative statics of I^* .

kS_0 , the expected rents of the manager, increase with k since the expected value of the pledgeable income, decreases with respect to k . ■

A large μ entails that the project has a large NPV, while large σ and k imply a severe agency problem. Not surprisingly, in line with the one-period version of this model (see Holmstrom and Tirole 1997), we find that the maximal possible investment of the firm increases with respect to the profitability of the project and decreases with respect to the severity of the agency problem. When the moral hazard problem is important, the firm starts out very illiquid, with a large amount of intangible assets. This is reminiscent of the balance sheet of technology start-ups, for which the assumption of unverifiable effort is highly relevant.

8 Appendix

Proof of Proposition 1. Let T be the Bellman operator associated to (6)–(9), and let $v \in C_b(\mathbb{R}_+)$. Note that the mapping $w \mapsto v(w) - (\rho - r)w/(1 + \rho)$ is coercive as $v \in C_b(\mathbb{R}_+)$ and $\rho > r$. Accordingly, let M_v be the maximum value of this function, and w_v the greatest point at which it reaches its maximum. Setting $x = w_0 = w_1 = 0$, we obtain that $Tv \geq 0$, so Tv is bounded below. Similarly, $Tv \leq \bar{R} + M_v/(1 + r)$, so Tv is bounded above. It is clear that for any value of w , there is no loss of generality in restricting w_0 and w_1 to be in $[0, w_v]$. Hence Berge's maximum theorem applies, and thus T maps $C_b(\mathbb{R}_+)$ into itself. Using Blackwell's sufficiency conditions, it is immediate to check that T is a contraction, and thus it has a unique fixed point $V \in C_b(\mathbb{R}_+)$ by the contraction mapping theorem.

We now prove that V is increasing. Since the set of continuous, bounded and increasing functions is a closed subset of $C_b(\mathbb{R}_+)$, it is sufficient to prove that T maps this set into itself. Specifically, let $v \in C_b(\mathbb{R}_+)$ be increasing, and let $w' \geq w \geq 0$. Assume that (x, w_0, w_1) is an optimal choice in the program that defines $Tv(w)$. Since $w' \geq w$, it follows from (7)–(9) that (x, w_0, w_1) is a feasible choice in the program that defines $Tv(w')$, and yields the same value as

$Tv(w)$. Hence $Tv(w') \geq Tv(w)$, which implies the result. That $V(0) = 0$ follows directly from (6) together with the fact that $w = 0$ implies $x = 0$ because of (7).

We now prove that V is concave. To do so, we decompose (6)–(9) into two subproblems. First, for any $w \geq pB/\Delta p$, consider the problem of maximizing the value conditional on continuation:

$$T^cV(w) = \max \left\{ \bar{R} + \frac{pV(w_1) + (1-p)V(w_0)}{1+r} - \frac{(\rho-r)[pw_1 + (1-p)w_0]}{(1+r)(1+\rho)} \right\}, \quad (25)$$

subject to:

$$w \geq \frac{w_0}{1+\rho} + \frac{pB}{\Delta p}, \quad (26)$$

$$w \geq \frac{pw_1 + (1-p)w_0}{1+\rho}, \quad (27)$$

and:

$$(w_0, w_1) \in \mathbb{R}_+^2. \quad (28)$$

Then the value function V , taking into account the possibility of liquidation, can be written as:

$$V(w) = \max\{xT^cV(w^c)\} \quad (29)$$

for each $w \geq 0$, subject to:

$$w = (1-x)l + xw^c \quad (30)$$

and:

$$(x, l, w^c) \in [0, 1] \times \mathbb{R}_+ \times [pB/\Delta p, \infty). \quad (31)$$

By the same argument used to show that V is increasing over \mathbb{R}_+ , it follows that T^cV is increasing over $[pB/\Delta p, \infty)$. Moreover, it is concave over this interval. Indeed, let $w, w' \geq pB/\Delta p$, $\lambda \in [0, 1]$ and $w_\lambda = \lambda w + (1-\lambda)w'$. Assume that (w_0, w_1) is optimal in the program that defines $T^cV(w)$, that (w'_0, w'_1) is optimal in the program that defines $T^cV(w')$, and let $(w_{0\lambda}, w_{1\lambda}) = \lambda(w_0, w_1) + (1-\lambda)(w'_0, w'_1)$. Since the constraints (26)–(28) are linear, it follows that $(w_{0\lambda}, w_{1\lambda})$ is a feasible choice in the program that defines $T^cV(w_\lambda)$. Since V is concave, it follows that $T^cV(w_\lambda) \geq \lambda T^cV(w) + (1-\lambda)T^cV(w')$, and thus T^cV is concave. Next, let \hat{w} be the smallest point at which the coercive mapping $w \mapsto V(w) - (\rho-r)w/(1+\rho)$ reaches its maximum, and let $w^* = \hat{w}/(1+\rho) + pB/\Delta p$. Then, for any $w \geq w^*$, (\hat{w}, \hat{w}) is a feasible choice

in the program that defines $T^cV(w)$, and it yields the maximum utility in (6) and (25), namely $\bar{R} + [V(\hat{w}) - (\rho - r)\hat{w}/(1 + \rho)]/(1 + r)$, which implies that T^cV is constant and coincides with V over $[w^*, \infty)$. Consider now the problem (29)–(31). Note first that since T^cV is increasing, it is optimal to set $l = 0$ for each $w \in \mathbb{R}_+$. It then follows that (29)–(31) can be rewritten as:

$$V(w) = \max \left\{ \frac{T^cV(w^c)}{w^c} \right\} w \quad (32)$$

for each $w \geq 0$, subject to:

$$w^c \geq \max\{pB/\Delta p, w\}. \quad (33)$$

Since T^cV is continuous over $[pB/\Delta p, \infty)$ and constant over $[w^*, \infty)$, and since $w^* > 0$ by construction, it is clear that the mapping $w^c \mapsto T^cV(w^c)/w^c$ reaches its maximum in $[pB/\Delta p, w^*]$. Moreover, since T^cV is concave, the set $\arg \max_{w^c \in [pB/\Delta p, w^*]} \{T^cV(w^c)/w^c\}$ is an interval $[\underline{w}^c, \bar{w}^c]$, possibly reduced to a point, and the mapping $w^c \mapsto T^cV(w^c)/w^c$ is increasing over $[pB/\Delta p, \underline{w}^c]$ and decreasing over $[\bar{w}^c, w^*]$. It follows from (32)–(33) that:

$$V(w) = \begin{cases} \max_{w^c \in [pB/\Delta p, w^*]} \{T^cV(w^c)/w^c\} w & \text{if } w \leq \bar{w}^c, \\ T^cV(w) & \text{if } w \geq \bar{w}^c, \end{cases}$$

and V is concave by construction.

Finally, suppose that for some $w < w' < w^*$, one has $V(w) = V(w')$. Then, by concavity of V , one must have $V(w) = V(w') = V(w^*)$. Therefore, the optimal choice in (6)–(9) given w must be $(1, \hat{w}, \hat{w})$, which violates (7), a contradiction. This implies that V is strictly increasing over $[0, w^*)$, as claimed.

Proof of Proposition 2. By definition of \hat{w} and the characterization of V given in Proposition 1, one clearly has $w^* \geq \hat{w}$. Suppose that $\hat{w} = w^* = (1 + \rho)pB/(\rho\Delta p)$. For any $\varepsilon > 0$ close enough to 0, the contract $(x, l, u_0, u_1, w_0, w_1) = (1, 0, 0, B/\Delta p, w^* - (1 + \rho)\varepsilon, w^* - (1 + \rho)\varepsilon)$ satisfies (2)–(5) and delivers a utility $w^* - \varepsilon$ to the manager. By (6), one must then have:

$$V(w^* - \varepsilon) \geq \bar{R} + \frac{V(w^* - (1 + \rho)\varepsilon)}{1 + r} - \frac{(\rho - r)[w^* - (1 + \rho)\varepsilon]}{(1 + r)(1 + \rho)}. \quad (34)$$

Moreover, since $\hat{w} = w^*$, one has:

$$V(w^*) = \bar{R} + \frac{V(w^*)}{1 + r} - \frac{(\rho - r)w^*}{(1 + r)(1 + \rho)}, \quad (35)$$

so rearranging (34) yields:

$$\frac{1 + \rho}{1 + r} \left[\frac{V(w^*) - V(w^* - (1 + \rho)\varepsilon)}{(1 + \rho)\varepsilon} \right] - \frac{V(w^*) - V(w^* - \varepsilon)}{\varepsilon} \geq \frac{\rho - r}{1 + r}.$$

Taking limits as ε goes to 0, and using the fact that $\rho > r$, it follows that:

$$V'_-(w^*) \geq 1.$$

Since V is concave and $V(0) = 0$, one thus gets that:

$$V(w^*) \geq w^*.$$

However, using the fact that $\bar{R} < pB/\Delta p = \rho w^*/(1 + \rho)$ whenever $\hat{w} = w^*$, it is immediate to check from (35) that $V(w^*) < w^*$, a contradiction. Hence $w^* > \hat{w}$, as claimed.

To prove the second inequality, notice that, since $w^* > \hat{w}$, $w^* = \hat{w}/(1 + \rho) + p/\Delta p$, and V is strictly increasing over $[0, w^*]$,

$$V(w^*) = \bar{R} + \frac{V(\hat{w})}{1 + r} - \frac{(\rho - r)\hat{w}}{(1 + r)(1 + \rho)} < \bar{R} + \frac{V(w^*)}{1 + r} - \frac{(\rho - r)w^*}{(1 + r)} + \frac{(\rho - r)pB}{(1 + r)\Delta p}.$$

Since $V(w^*) \geq 0$, it follows that:

$$\frac{(\rho - r)w^*}{(1 + r)} < \bar{R} + \frac{(\rho - r)pB}{(1 + r)\Delta p},$$

which implies the result as $\rho > r$.

Proof of Proposition 3. We simply have to check that $TV = V$. Since the slope of V is less or equal than $(\rho - r)/(1 + \rho)$, it follows that setting $w_0 = w_1 = 0$ is optimal for each $w \geq 0$. For $w \geq pB/\Delta p$, it is optimal to set $x = 1$ and thus $TV(w) = \bar{R} = V(w)$. For $w < pB/\Delta p$, it is optimal to set $x = w/(pB/\Delta p)$ and thus $TV(w) = [\bar{R}/(pB/\Delta p)]w = V(w)$ as well. It is immediate to check that $\hat{w} = 0$ and $w^* = pB/\Delta p$.

Conversely, suppose that $\hat{w} = 0$. Then necessarily $w^* = pB/\Delta p$, which implies by (7) that the optimal contract involves positive probabilities of liquidation for $w \in [0, w^*)$, and thus that V is linear over this interval. Moreover, $\hat{w} = 0$ implies that the slope of V over $[0, w^*]$ is less or equal than $(\rho - r)/(1 + \rho)$, and that $V(w^*) = \bar{R}$. Since $w^* = pB/\Delta p$, (10) follows.