Preliminary Notes on Decentralized Auctions of Complements

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Started: March 23, 2004; this version: May 10, 2004

§1. Introduction

This note was motivated by a feature of decentralized markets: usually there is no central coordination on the selling mechanisms of items owned by different individuals. Specifically, I analyze separate auctions of several items possibly followed by resale among bidders. These items are complements for some bidders and are initially owned by different individuals. Without central coordination, Vickrey-Clarke-Groves (VCG) mechanisms are not used, and the initial owners auction off their items separately and simultaneously. (An owner of multiple items may accept package-bids.) I analyze these auctions in both the case where resale is prohibited and the case where resale is allowed. As these auctions do not guarantee efficiency, resale may occur. At the resale stage, if a bidder has acquired all the items, he gets to pick and commit to a mechanism for their possible resale. With resale taken into account, the initial round of auctions turns out to be easier to analyze than the case where resale is assumed impossible. When resale is allowed, it makes no difference, in efficiency and in revenue, whether two items, initially owned by a single entity, are auctioned off through separate auctions or a package auction. While the equilibria in the case that prohibits resale require players to reason interactively, the equilibria in the case that allows resale simplify players' problems into decision-theoretic ones.

There has been no other paper on auctions of multiple heterogeneous goods with resale allowed. There are a few papers on multiple-object auction with possible synergies among objects and with the implicit assumption that resale is banned. These are close to §3 of this note. Krishna and Rosenthal [6] analyze sealed-bid simultaneous second-price auctions and a variant where a package-bidder submits a sealed bid for the package only once and then, if this bid fails to win, the auctioneer runs simultaneous second-price sealed-bid auctions. Rosenthal and Wang [10] analyze sealed-bid first-price simultaneous auctions. My analysis on the no-resale case (§3) differs from these papers by considering the dynamic aspect of ascending-bid auctions. Albano, Germano and Lovo [1] analyze a simultaneous ascending-bid auction with two items whose values are uniformly distributed. The analysis in $3.1 of my note differs from it by considering three items (extendable to n items) with bidders desiring different packages and general distributions for bidders' valuations. Menezes and Monteiro [8] analyze sequential second-price auctions. Ausubel and Milgrom [2] design and analyze a revelation mechanism where proxies submit package-bids for bidders. I model the interaction between auction and resale by extending the single-good model in Zheng [11].

§2. The model

There are three items, A, B, and C. There are three kinds of bidders. A *global* bidder views all three items as complements, a *regional* bidder views items A and B as complements, and a *local* bidder values only one of the items. The following table lists their valuations. Here $(\alpha\beta\gamma, i)$ denotes the ith global bidder, $(\alpha\beta, i)$ denotes the ith regional bidder, (α, i) denotes the ith local bidder who values only item A, and likewise for (β, i) and (γ, i).

	none	A	B	C	A & B	A & B & C
$(\alpha\beta\gamma, i)$	0	a	b	c	$a + b$	$a + b + c + t^i_{\alpha\beta\gamma}$
$(\alpha\beta, i)$	0	a	b	0	$a + b + t^i_{\alpha\beta}$	$a + b + t^i_{\alpha\beta}$
(α, i)	0	$a + t^i_\alpha$	0	0	$a + t^i_\alpha$	$a + t^i_\alpha$
(β, i)	0	0	$b + t^i_\beta$	0	$b + t^i_\beta$	$b + t^i_\beta$
(γ, i)	0	0	0	$c + t^i_\gamma$	0	$c + t^i_\gamma$

The parameters a, b, and c are positive and commonly known. For each bidder (k, i) ($k \in \{\alpha\beta\gamma, \alpha\beta, \alpha, \beta, \gamma\}$), t^i_k is a one-dimension random variable whose realized value is the private information of the bidder, and it is independently drawn from a distribution F_k (hence bidders of the same kind are identically distributed), with density f_k and support $[0, \bar{t}_k]$.

There are N periods, with no discounting, and N is exogenous and much larger than the number of bidders. In period one, the items are auctioned off via mechanisms described after this paragraph. In period two, resale among bidders is allowed. If a bidder has won all items in period one, he can pick any transparent mechanism (explained in Zheng [11, §2.1]) and commit to it for possible resale. If items are sold to different bidders, one of the winners is randomly selected to pick a resale mechanism; if no other winner vetoes it, the mechanism is implemented; else the mechanism is not implemented and every winner commits to a resale price for the item he currently owns. The probability with which a winner is selected to pick a resale mechanism is proportional to the number of items he currently owns. If a resale mechanism results in no-sale or if period N reached, the game ends; else in the next period a winner is chosen to pick a transparent resale mechanism, as in the current period.

In period one, the items are auctioned off either by (i) simultaneous ascending bid auctions or by (ii) a package auction of items A and B that goes simultaneously with an English auction of item C. I do not assume that the initial auctions to constitute a VCG mechanism, because individual sellers in decentralized markets may lack the coordination device to run a centralized mechanism such as VCG. (Specifically, the initial seller of item C is different from the initial owner(s) of items A and B. If mechanism (ii) is in use then items A and B are initially owned by the same entity, otherwise it may be infeasible to sell A and B via a package auction). I assume each auction to be in the open-outcry ascending-bid format in order to ensure transparency for bidders, a usual feature of decentralized markets. I assume different auctions to be operated simultaneously in order to capture the interactions among different sectors of an economy without artificially ranking one sector over another.

§2.1. Simultaneous ascending bid auctions

In this case, the items are auctioned off in period one via three separate English auctions that are held simultaneously. The price p_A (or p_B, p_C) for item A (or B, C) starts at the level a (or b, c) and rises continuously at a speed equal to $\dot{p}_A$ (or $\dot{p}_B$, $\dot{p}_C$) until all but one bidder have quit the auction. Reentry is not allowed. Bidding actions are commonly observed.

§2.2. Package auction

In this case, item C is auctioned off in period one via an English auction that bans reentry, with the price starting at c and rising at a speed $\dot{p}_C$. At the same time, items A and B are auctioned off via a package auction that also bans reentry:

1. A bidder may bid on at most one of the three packages: $\{\mathrm{A}\}$, $\{\mathrm{B}\}$, and $\{\mathrm{A}, \mathrm{B}\}$.

2. The price p_A (or p_B, p_{AB}) for package $\{\mathrm{A}\}$ (or $\{\mathrm{B}\}$, $\{\mathrm{A}, \mathrm{B}\}$) starts at a (or b, $a+b$). If all bidders are active, p_A and p_B rise at the same speed and p_{AB} rises at twice that speed. Hence $p_{AB} = p_A + p_B$ when all three are active.

3. If all bidders for package $\{\mathrm{A}, \mathrm{B}\}$ have quit,
    a. if the highest bidder for $\{\mathrm{A}\}$ (or $\{\mathrm{B}\}$) has been determined, sell item A (or B) to the highest bidder at the current price p_A (or p_B);
    b. if the highest bidder for $\{\mathrm{A}\}$ (or $\{\mathrm{B}\}$) has not been determined, continue the process as an English auction until the item is sold.

4. If the highest bidder for $\{\mathrm{A}\}$ has quit at price p_A while there are still active bidders for $\{\mathrm{B}\}$ and $\{\mathrm{A}, \mathrm{B}\}$, stop raising p_A, and raise p_B and p_{AB} at the same speed; hence we still have $p_{AB} = p_A + p_B$.
    a. If subsequently the highest bidder for $\{\mathrm{B}\}$ quits at price p_B,
        i. if the highest bidder for $\{\mathrm{A}, \mathrm{B}\}$ has been determined, sell both items to him at price $p_A + p_B$;
        ii. if the highest bidder for $\{\mathrm{A}, \mathrm{B}\}$ has not been determined, continue the process as an English auction for the whole package $\{\mathrm{A}, \mathrm{B}\}$ until it is sold to a bidder.
    b. If subsequently the highest bidder for $\{\mathrm{A}, \mathrm{B}\}$ has quit at price p_{AB}, sell item A to the highest bidder for $\{\mathrm{A}\}$ at price p_A and—
        i. if the highest bidder for $\{\mathrm{B}\}$ has been determined, sell B to him at price $p_{AB} - p_A$ $(= p_B)$;
        ii. if the highest bidder for $\{\mathrm{B}\}$ has not been determined, continue the process as an English auction for item B until it is sold.

5. If the highest bidder for $\{\mathrm{B}\}$ has quit at price p_B while there are active bidders for $\{\mathrm{A}\}$ and $\{\mathrm{A}, \mathrm{B}\}$, do the same thing as in provision 4 with the roles of A and B switched.

§2.3. Price normalization

In the aforementioned auctions, bids start from the levels a (for item A), b (for item B), c (for C), $a+b$ (for $\{\mathrm{A}, \mathrm{B}\}$). Thus, for simplicity in notations, we denote:

$$p_A := p_A - a; \quad p_B := p_B - b; \quad p_C := p_C - c; \quad p_{AB} := p_{AB} - a - b.$$

In the sequel, "the price of A is equal to p_A" means "the price of A is equal to $p_A + a$." Analogous interpretations apply to p_B, p_C, and p_{AB}.

§3. Equilibria when resale is prohibited

In existing literature, several complete-information examples have been constructed where the two mechanisms considered here suffer inefficiency in the form of exposure and threshold problems (e.g., Ausubel and Milgrom [2], Bikhchandani, de Vries, Schummer and Vohra [3], and DeMartini, Kwasnica, Ledyard, and Porter [4]). But there has not been an analysis based on asymmetric information. This section fills that gap.

§3.1. The exposure problem of simultaneous auctions

Let us analyze the simultaneous ascending bid auction under the assumption that resale is impossible. Obviously a local bidder's dominant strategy is to bid for his desired item up to its true value. This is not so for a regional or global bidder, because the bidder takes into account the exposure problem that he may end with buying an item at a price above its standalone value and failing to acquire its complement.

Consider any equilibrium where local bidders play the above dominant strategy. I shall derive non-local bidders' best replies to this equilibrium. Coupled with local bidders' sincere bidding, these best replies constitute an equilibrium. For every $k \in \{\alpha, \beta, \gamma, \alpha\beta\}$, denote:

$$\begin{aligned} n_k &:= \text{the number of bidders of kind } k \text{ who have not quit;} \\ t_k^{(j)} &:= \text{the } j\text{th highest realized type among bidders of kind } k; \\ z^+ &:= \max\{0, z\}. \end{aligned}$$

If $g(x,y)$ and $h(z)$ are real functions of the variables x, y, and z, let $\mathrm{E}[f(\mathbf{x}, y) \mid h(\mathbf{z}) \geq 0]$ denote the expected value of $g(x,y)$, with the random variables **boldfaced** in the bracket, conditional on $h(z) \geq 0$. The interpretation for the next notations will be clear soon.

$$v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) := \begin{cases} t^i_{\alpha\beta} - p_B & \text{if } n_\beta = 0 \\ \mathrm{E}\left[\left(t^i_{\alpha\beta} - \mathbf{t}^{(1)}_\beta\right)^+ \middle| \mathbf{t}^{(n_\beta)}_\beta \geq p_B\right] & \text{if } n_\beta \geq 1; \end{cases} \tag{1}$$

$$v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) := \begin{cases} t^i_{\alpha\beta} - p_A & \text{if } n_\alpha = 0 \\ \mathrm{E}\left[\left(t^i_{\alpha\beta} - \mathbf{t}^{(1)}_\alpha\right)^+ \middle| \mathbf{t}^{(n_\alpha)}_\alpha \geq p_A\right] & \text{if } n_\alpha \geq 1. \end{cases} \tag{2}$$

Lemma 1 *Given any current prices (p_A, p_B) and current numbers (n_α, n_β) of remaining local bidders, the best reply from a regional bidder $(\alpha\beta, i)$ of type $t^i_{\alpha\beta}$ is:*

1. *If neither A nor B has had a winner, continue bidding for both items if $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) > p_A$ and $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) > p_B$, and quit from both auctions once the prices are high enough for at least one of these inequalities to fail.*
2. *If item A or B has been won by someone else, quit from both auctions immediately.*
3. *If item A (or B) has been won by this bidder, continue bidding for item B (or A) until its (normalized) price reaches $t^i_{\alpha\beta}$ (or $t^i_{\alpha\beta}$).*

Proof First, we make two observations:

Remark 1 *If an item in a non-local bidder's desired package ($\{A, B\}$ or $\{A, B, C\}$) has been won by someone else, or if the bidder has quit bidding for an item in his desired package, then the bidder immediately quits bidding for the other item(s).*

The reason for this remark is obvious: The price for an item say A is for sure higher than its standalone value a for the bidder, and resale is assumed impossible in this section.

Remark 2 *If a non-local bidder, with realized type t^i_k, has won all items in his desired package except item A (or B, C), then he continues bidding for A (or B, C) until its price reaches t^i_k (or t^i_k, t^i_k).*

The reason is simply that the payment for the already acquired items is sunk, since resale is assumed impossible in this subsection.

Now consider a regional bidder $(\alpha\beta, i)$. wins item A. By the above remarks, contingency plans 2 and 3 in the lemma are optimal for the bidder. We need only to consider contingency 1, where the bidder is bidding for both A and B and neither has had a winner. Suppose the current prices are (p_A, p_B) and the numbers of remaining local bidders are (n_α, n_β). We claim that the expected value of the profit from acquiring item A at this current instant is equal to

$$v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) - p_A.$$

To see that, consider the case $n_\beta > 0$. If the bidder has won item A, all global bidders and all other regional bidders quit immediately (Remark 1) and hence the local bidders (β, j) are the only other contenders for item B, so bidder $(\alpha\beta, i)$ wins B if $t^i_{\alpha\beta} > t^{(1)}_\beta$ and loses B if the inequality is reversed (Remark 2). Thus, the expected profit from acquiring A at this current instant is equal to the second branch of Eq. (1) minus p_A. Next consider the other case, $n_\beta = 0$ (all local bidders for B have quit). Then winning item A implies winning item B immediately (Remark 1) and so the profit of buying A now is equal to the first branch of Eq. (1) minus p_A. Hence the claim is true. Analogously, the expected value of the profit from buying item B at this current instant is equal to

$$v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) - p_B.$$

To complete the proof, we make a straightforward observation:

Remark 3 *For any n_β, $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta)$ is strictly decreasing in p_B and is increasing in $t^i_{\alpha\beta}$ and strictly so when $t^i_{\alpha\beta} \geq p_B$. For any n_α, $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha)$ is strictly decreasing in p_A and is increasing in $t^i_{\alpha\beta}$ and strictly so when $t^i_{\alpha\beta} \geq p_A$.*

There are only three cases when the regional bidder is bidding for both A and B and neither has had a winner:

1. If $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) > p_A$ and $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) > p_B$, then it is dominated to quit from one item and continue bidding for the other (previous analysis), and it is also dominated to

quit on both items, because doing so gives only zero payoff, while continuation ensures a positive expected payoff. Hence the bidder continues bidding for both items.

2. If prices have risen to a threshold such that $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) = p_A$ while $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) \geq p_B$, then from now on $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) < p_A$ will be true (Remark 3) and hence it is dominated to continue bidding for A even if the bidder plans to continue with B. Hence the bidder quits at least from item A, which implies that he quits from item B immediately (Remark 1). Thus, the bidder quits from both auctions.
3. If prices have risen to a threshold such that $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) = p_B$ while $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) \geq p_B$, the bidder quits both auctions. The reasoning is the same as the previous case.

The proof of the lemma is complete. ■

More notations are needed before deriving a global bidder's best reply. Consider the *basic case* of having exactly one α-local bidder, one β-local bidder, one $\alpha\beta$-regional bidder, and no global bidder. Let $(t_\alpha, t_\beta, t_{\alpha\beta})$ denote their realized types. From Lemma 1, we can derive the equilibrium allocation of items A and B. Figure 1 depicts this allocation from the regional bidder's viewpoint. In this figure, the ray OD represents the trajectory of the price vector when both auctions are ongoing (recall that the price for item j rises at the exogenous

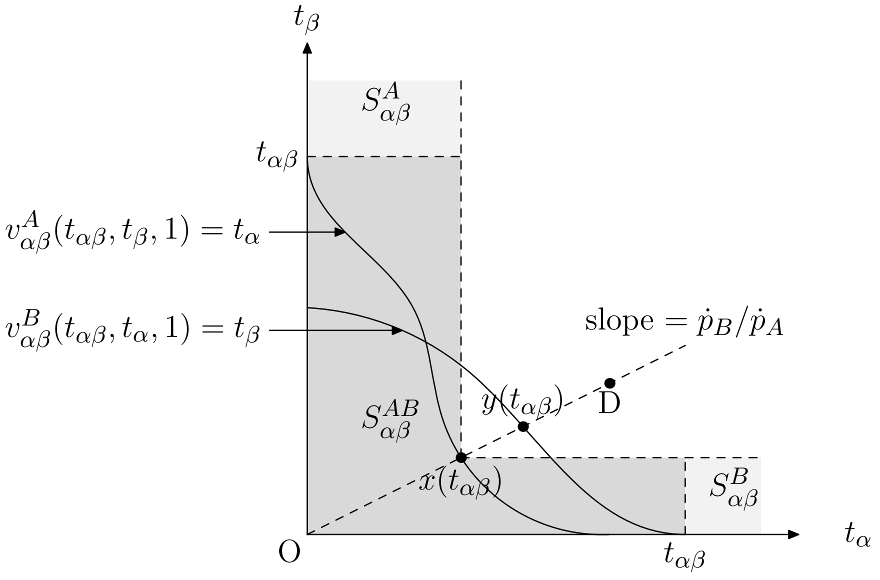

Figure 1: Simultaneous auctions in the basic case

speed $\dot{p}_j$). The regional bidder wins both items on the dark area $S^{AB}_{\alpha\beta}$, wins item A and not B on the light gray area $S^{A}_{\alpha\beta}$ on the upper part of the diagram, and wins item B and not A on the light gray area $S^{B}_{\alpha\beta}$ on the right part of the diagram. On the other (uncolored) area, item A goes to local bidder α and item B goes to local bidder β. (To understand the picture, we just need to observe a fact: if (p_A, p_B) lies below the curve $v^A_{\alpha\beta}(t_{\alpha\beta}, t_\beta, 1) = t_\alpha$, then $v^A_{\alpha\beta}(t_{\alpha\beta}, p_B, 1) > p_A$ by Remark 3, hence the regional bidder would continue bidding for A if he is still bidding for B; analogously, if (p_A, p_B) lies below the curve $v^B_{\alpha\beta}(t_{\alpha\beta}, t_\alpha, 1) = t_\beta$, then $v^B_{\alpha\beta}(t_{\alpha\beta}, p_A, 1) > p_B$.)

Denote $x(t_{\alpha\beta}) := (x_\alpha(t_{\alpha\beta}), x_\beta(t_{\alpha\beta}))$ for the intersection point between ray OD and curve $v^A_{\alpha\beta}(t_{\alpha\beta}, t_\beta, 1) = t_\alpha$, and $y(t_{\alpha\beta}) := (y_\alpha(t_{\alpha\beta}), y_\beta(t_{\alpha\beta}))$ for the intersection point between ray OD and curve $v^B_{\alpha\beta}(t_{\alpha\beta}, t_\alpha, 1) = t_\beta$, as labeled in Figure 1. (By properties of $v^A_{\alpha\beta}$ and $v^B_{\alpha\beta}$, these points exist and are uniquely determined by $t_{\alpha\beta}$ and the exogenous $\dot{p}_B/\dot{p}_A$.) Let

$$z(t_{\alpha\beta}) := (z_\alpha(t_{\alpha\beta}), z_\beta(t_{\alpha\beta})) := \left(\min\left\{x_\alpha(t_{\alpha\beta}), y_\alpha(t_{\alpha\beta})\right\}, \min\left\{x_\beta(t_{\alpha\beta}), y_\beta(t_{\alpha\beta})\right\}\right). \tag{3}$$

Note that either $z(t_{\alpha\beta}) = x(t_{\alpha\beta})$ or $z(t_{\alpha\beta}) = y(t_{\alpha\beta})$.

In the sequel, $S^{AB}_{\alpha\beta}$, $S^{A}_{\alpha\beta}$, and $S^{B}_{\alpha\beta}$ denote the sets of (t_α, t_β) as labeled in Figure 1. Analogously, we define the notations $S^{BC}_{\beta\gamma}$, $S^{B}_{\beta\gamma}$, and $S^{C}_{\beta\gamma}$, with (β, γ, B, C) taking the role of (α, β, A, B). The notations $S^{B}_{\beta\gamma}$ and so forth are defined likewise. Denote 1_S for the indicator function for a set S. The interpretation for the following notations will be clear soon.

$$\begin{aligned}
u_A(t_{\alpha\beta\gamma}, t_\beta, t_\gamma) \;&:=\; (t_{\alpha\beta\gamma} - t_\beta - t_\gamma)\, 1_{S^{BC}_{\beta\gamma}}(t_\beta, t_\gamma) \\
&\quad + (t_{\alpha\beta\gamma} - t_\beta)\, 1_{S^{B}_{\beta\gamma}}(t_\beta, t_\gamma) + (t_{\alpha\beta\gamma} - t_\gamma)\, 1_{S^{C}_{\beta\gamma}}(t_\beta, t_\gamma); \\
u_B(t_{\alpha\beta\gamma}, t_\alpha, t_\gamma) \;&:=\; (t_{\alpha\beta\gamma} - t_\alpha - t_\gamma)\, 1_{S^{AC}_{\alpha\gamma}}(t_\alpha, t_\gamma) \\
&\quad + (t_{\alpha\beta\gamma} - t_\alpha)\, 1_{S^{A}_{\alpha\gamma}}(t_\alpha, t_\gamma) + (t_{\alpha\beta\gamma} - t_\gamma)\, 1_{S^{C}_{\alpha\gamma}}(t_\alpha, t_\gamma); \\
u_C(t_{\alpha\beta\gamma}, t_\alpha, t_\beta) \;&:=\; (t_{\alpha\beta\gamma} - t_\alpha - t_\beta)\, 1_{S^{AB}_{\alpha\beta}}(t_\alpha, t_\beta) \\
&\quad + (t_{\alpha\beta\gamma} - t_\alpha)\, 1_{S^{A}_{\alpha\beta}}(t_\alpha, t_\beta) + (t_{\alpha\beta\gamma} - t_\beta)\, 1_{S^{B}_{\alpha\beta}}(t_\alpha, t_\beta);
\end{aligned} \tag{4}$$

$$\begin{aligned}
w_\alpha(t_{\alpha\beta\gamma}, t_{\alpha\beta}, t_\alpha) \;&:=\; 1_{t_{\alpha\beta\gamma} > t_{\alpha\beta}}(t_{\alpha\beta}) \left[\begin{array}{l} 1_{t_\alpha \geq y_\alpha(t_{\alpha\beta})}(t_{\alpha\beta}, t_\alpha)\left((t_{\alpha\beta\gamma} - t_\alpha)^+ - y_\beta(t_{\alpha\beta})\right) \\ +1_{t_\alpha < y_\alpha(t_{\alpha\beta})}(t_{\alpha\beta}, t_\alpha)(t_{\alpha\beta\gamma} - t_{\alpha\beta}) \end{array} \right]; \\
w_\beta(t_{\alpha\beta\gamma}, t_{\alpha\beta}, t_\beta) \;&:=\; 1_{t_{\alpha\beta\gamma} > t_{\alpha\beta}}(t_{\alpha\beta}) \left[\begin{array}{l} 1_{t_\beta \geq x_\beta(t_{\alpha\beta})}(t_{\alpha\beta}, t_\beta)\left((t_{\alpha\beta\gamma} - t_\beta)^+ - x_\alpha(t_{\alpha\beta})\right) \\ +1_{t_\beta < x_\beta(t_{\alpha\beta})}(t_{\alpha\beta}, t_\alpha)(t_{\alpha\beta\gamma} - t_{\alpha\beta}) \end{array} \right];
\end{aligned} \tag{5}$$

$$
\begin{aligned}
w(t_{\alpha\beta\gamma}, t_{\alpha\beta}, t_\alpha, t_\beta) \quad := \quad & u_C(t_{\alpha\beta\gamma}, t_\alpha, t_\beta) 1_{z(t_{\alpha\beta}) \;\leqq\; (t_\alpha, t_\beta)}(t_{\alpha\beta}, t_\alpha, t_\beta) \\
& + w_\alpha(t_{\alpha\beta\gamma}, t_{\alpha\beta}, t_\alpha) 1_{z_\alpha(t_{\alpha\beta}) \le t_\alpha, z_\beta(t_{\alpha\beta}) > t_\beta}(t_{\alpha\beta}, t_\alpha, t_\beta) \\
& + w_\beta(t_{\alpha\beta\gamma}, t_{\alpha\beta}, t_\beta) 1_{z_\alpha(t_{\alpha\beta}) > t_\alpha, z_\beta(t_{\alpha\beta}) \le t_\beta}(t_{\alpha\beta}, t_\alpha, t_\beta) \\
& + (t_{\alpha\beta\gamma} - t_{\alpha\beta})^+ 1_{z(t_{\alpha\beta}) > (t_\alpha, t_\beta)}(t_{\alpha\beta}, t_\alpha, t_\beta);
\end{aligned}
\tag{6}
$$

$$
v^A_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_B, p_C, n_\beta, n_\gamma) \quad := \quad
\begin{cases}
\mathrm{E}\left[u_A(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_\beta, \mathbf{t}^{(1)}_\gamma) \,\middle|\, \begin{matrix} \mathbf{t}^{(n_\beta)}_\beta \ge p_B \\ \mathbf{t}^{(n_\gamma)}_\gamma \ge p_C \end{matrix} \right] & \text{if } n_\beta n_\gamma > 0 \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\gamma \right)^+ - p_B \,\middle|\, \mathbf{t}^{(n_\gamma)}_\gamma \ge p_C \right] & \text{if } n_\beta = 0 < n_\gamma \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\beta \right)^+ - p_C \,\middle|\, \mathbf{t}^{(n_\beta)}_\beta \ge p_B \right] & \text{if } n_\beta > 0 = n_\gamma \\
t^i_{\alpha\beta\gamma} - p_B - p_C & \text{if } n_\beta = n_\gamma = 0;
\end{cases}
\tag{7}
$$

$$
v^B_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_A, p_C, n_\alpha, n_\gamma) \quad := \quad
\begin{cases}
\mathrm{E}\left[u_B(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_\alpha, \mathbf{t}^{(1)}_\gamma) \,\middle|\, \begin{matrix} \mathbf{t}^{(n_\alpha)}_\alpha \ge p_A \\ \mathbf{t}^{(n_\gamma)}_\gamma \ge p_C \end{matrix} \right] & \text{if } n_\alpha n_\gamma > 0 \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\gamma \right)^+ - p_A \,\middle|\, \mathbf{t}^{(n_\gamma)}_\gamma \ge p_C \right] & \text{if } n_\alpha = 0 < n_\gamma \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\alpha \right)^+ - p_C \,\middle|\, \mathbf{t}^{(n_\alpha)}_\alpha \ge p_A \right] & \text{if } n_\alpha > 0 = n_\gamma \\
t^i_{\alpha\beta\gamma} - p_A - p_C & \text{if } n_\alpha = n_\gamma = 0;
\end{cases}
\tag{8}
$$

$$
v^C_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_A, p_B, n_\alpha, n_\beta) \quad := \quad
\begin{cases}
\mathrm{E}\left[u_C(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_\alpha, \mathbf{t}^{(1)}_\beta) \,\middle|\, \begin{matrix} \mathbf{t}^{(n_\alpha)}_\alpha \ge p_A \\ \mathbf{t}^{(n_\beta)}_\beta \ge p_B \end{matrix} \right] & \text{if } n_\alpha n_\beta > 0 \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\beta \right)^+ - p_A \,\middle|\, \mathbf{t}^{(n_\beta)}_\gamma \ge p_B \right] & \text{if } n_\alpha = 0 < n_\beta \\
\mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_\alpha \right)^+ - p_B \,\middle|\, \mathbf{t}^{(n_\alpha)}_\alpha \ge p_A \right] & \text{if } n_\alpha > 0 = n_\beta \\
t^i_{\alpha\beta\gamma} - p_A - p_B & \text{if } n_\alpha = n_\beta = 0;
\end{cases}
\tag{9}
$$

$$\phi(t^i_{\alpha\beta\gamma}, p_A, p_B, n_{\alpha\beta}, n_\alpha, n_\beta) := \begin{cases} v^C_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_A, p_B, n_\alpha, n_\beta) & \text{if } n_{\alpha\beta} = 0 \\ \mathrm{E}\left[\left(t^i_{\alpha\beta\gamma} - \mathbf{t}^{(1)}_{\alpha\beta}\right)^+ \middle| \mathbf{t}^{(n_{\alpha\beta})}_{\alpha\beta} \geq p_A + p_B\right] & \text{if } \begin{cases} n_{\alpha\beta} > 0 \\ n_\alpha = n_\beta = 0 \end{cases} \\ \mathrm{E}\left[w_\alpha(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_{\alpha\beta}, \mathbf{t}^{(1)}_\alpha) \middle| \begin{array}{l} y\left(\mathbf{t}^{(n_{\alpha\beta})}_{\alpha\beta}\right) \geqq (p_A, p_B); \\ \mathbf{t}^{(n_\alpha)}_\alpha \geq p_A \end{array} \right] & \text{if } \begin{cases} n_{\alpha\beta} n_\alpha > 0 \\ n_\beta = 0 \end{cases} \\ \mathrm{E}\left[w_\beta(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_{\alpha\beta}, \mathbf{t}^{(1)}_\beta) \middle| \begin{array}{l} x\left(\mathbf{t}^{(n_{\alpha\beta})}_{\alpha\beta}\right) \geqq (p_A, p_B); \\ \mathbf{t}^{(n_\beta)}_\beta \geq p_B \end{array} \right] & \text{if } \begin{cases} n_{\alpha\beta} n_\beta > 0 \\ n_\alpha = 0 \end{cases} \\ \mathrm{E}\left[w(t^i_{\alpha\beta\gamma}, \mathbf{t}^{(1)}_{\alpha\beta}, \mathbf{t}^{(1)}_\alpha, \mathbf{t}^{(1)}_\beta) \middle| \begin{array}{l} \mathbf{t}^{(n_\alpha)}_\alpha \geq p_A \\ \mathbf{t}^{(n_\beta)}_\beta \geq p_B \\ z\left(\mathbf{t}^{(1)}_{\alpha\beta}\right) \geqq (p_A, p_B) \end{array} \right] & \text{if } n_{\alpha\beta} n_\alpha n_\beta > 0 \end{cases} \tag{10}$$

Lemma 2 *Given any current prices (p_A, p_B, p_C) and current numbers $(n_\alpha, n_\beta, n_{\alpha\beta})$ of remaining non-global bidders, the best reply from a global bidder $(\alpha\beta\gamma, i)$ of type $t^i_{\alpha\beta\gamma}$ is:*

1. *If no item has had a winner, continue bidding for all items if $v^A_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_B, p_C, n_\beta, n_\gamma) > p_A$, $v^B_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_A, p_C, n_\alpha, n_\gamma) > p_B$, and $\phi(t^i_{\alpha\beta\gamma}, p_A, p_B, n_{\alpha\beta}, n_\alpha, n_\beta) > p_C$, and quit all auctions if one of these inequalities fails.*

2. *If an item has been won by someone else, quit from all auctions immediately.*

3. *If item k has been won by this bidder, continue bidding for the other two items l and m by following the best reply (Lemma 1) of a regional bidder whose desired package is $\{l, m\}$ and whose realized type is equal to $t^i_{\alpha\beta\gamma}$.*

Proof Contingency plans 2 and 3 are obvious, as in Lemma 1. We need only to examine contingency plan 1, when no item has had a winner and the global bidder remains active for all items. Let π_k denote the bidder's expected value of profits from buying item k at the current instant. I claim:

$$\pi_k = \begin{cases} v^A_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_B, p_C, n_\beta, n_\gamma) - p_A & \text{if } k = A \\ v^B_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_A, p_C, n_\alpha, n_\gamma) - p_B & \text{if } k = B \\ \phi(t^i_{\alpha\beta\gamma}, p_A, p_B, n_{\alpha\beta}, n_\alpha, n_\beta) - p_C & \text{if } k = C. \end{cases} \tag{11}$$

If (11) is true, the optimality of contingency plan 1 follows from the fact that the functions $v^A_{\alpha\beta\gamma}$, $v^B_{\alpha\beta\gamma}$, and ϕ are decreasing in, respectively, p_A, p_B, and p_C, by the same reasoning in Lemma 1. Hence it suffices to prove (11).

First, we prove the first branch of (11). If the global bidder wins item A, all non-local bidders quit immediately (Remark 1) and he continues as if he were a regional bidder desiring for $\{\text{B}, \text{C}\}$ with realized type $t^i_{\alpha\beta\gamma}$ (contingency plan 3). There are only four cases:

a. Some local bidders are active for items B and C ($n_\beta n_\gamma > 0$). Then the allocation of B and C is determined in the same way as depicted in Figure 1, with the symbols $(t_{\alpha\beta}, t_\alpha, t_\beta, A, B)$ in that figure replaced by $(t^i_{\alpha\beta\gamma}, t^{(1)}_\beta, t^{(1)}_\gamma, B, C)$. Thus, the global bidder's ex post profit from buying item A at its current price p_A is equal to $u_A(t^i_{\alpha\beta\gamma}, t^{(1)}_\beta, t^{(1)}_\gamma) - p_A$ (with u_A defined in (4)), and its expected value, as calculated in (7), is equal to the first branch of the right-hand side of (11).

b. All local β-bidders have quit and some γ-bidders are active ($n_\beta = 0 < n_\gamma$). Then winning item A means winning item B immediately at its current price p_B, because all non-local bidders quit all auctions once A goes to this bidder. After winning A and B, the global bidder continues bidding for C until its price reaches $t^i_{\alpha\beta\gamma}$ (Remark 2). Thus, π_A is equal to the second branch of the right-hand side of (7) minus p_A, i.e., $\pi_A = v^A_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_B, p_C, n_\beta, n_\gamma) - p_A$.

c. $n_\beta > 0 = n_\gamma$. This case is analogous to case (b), with (β, B) and (γ, C) switched.

d. $n_\beta = n_\gamma = 0$. Without any local bidders left, winning item A means winning the other items at their current prices (p_B, p_C), as the non-local bidders quit immediately. Hence $\pi_A = t^i_{\alpha\beta\gamma} - p_B - p_C - p_A = v^A_{\alpha\beta\gamma}(t^i_{\alpha\beta\gamma}, p_B, p_C, n_\beta, n_\gamma) - p_A$.

The second branch of (11) is similar to the first branch, since all non-local bidders quit immediately after the global bidder has won item B.

The third branch of (11) is more complex, because regional $\alpha\beta$-bidders (who desire only $\{\text{A}, \text{B}\}$) may stay active after the global bidder has won item C, and a regional bidder's maximum bid jumps (from one branch to another on the right-hand sides of Eqs. (1) and (2)) when all local bidders quit from an item. The easy case is $n_{\alpha\beta} = 0$. Here, with regional bidders already gone, this global bidder after winning C becomes the only non-local bidder (since other global bidders quit once he wins C by Remark 1). Then his profit is calculated as in the previous branches and is equal to the first branch of the right-hand side of (10) (defined in (9)). Let us consider the other case, $n_{\alpha\beta} > 0$, with four sub-cases:

a. $n_\alpha = n_\beta = 0$. Then after winning item C the global bidder is competing only with regional $\alpha\beta$-bidders (other global bidders quit by Remark 1). With all local bidders gone, a non-local bidder who wins an item wins the other item immediately (Remark 1). Thus, a regional bidder continues bidding for A and B until $p_A + p_B$ reaches his realized type. Hence the global bidder's ex post profit from buying item C right now is equal to $\left(t^i_{\alpha\beta\gamma} - t^{(1)}_{\alpha\beta}\right)^+ - p_C$, so the second branch of (10) implies (11) in this case.

b. $n_\alpha > 0 = n_\beta$. I claim that the global bidder's ex post profit from buying C is equal to $w_\alpha\left(t^i_{\alpha\beta\gamma}, t^{(1)}_{\alpha\beta}, t^{(1)}_\alpha\right) - p_C$. This claim implies that (11) follows from (10). To justify this claim, we examine Eq. (5), which defines w_α. The indicator function $1_{t_{\alpha\beta\gamma} > t_{\alpha\beta}}$ in (5) captures the fact that this global bidder would lose both items if his type does not exceed the types of all $\alpha\beta$-bidders, because this global bidder after winning C acts as a regional bidder (Remark 2) and a regional bidder's maximum bid is strictly increasing in the bidder's type (Remark 3). With $n_\alpha > 0 = n_\beta$, a regional bidder $(\alpha\beta, i)$ continues bidding for A and B as long as $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) > p_B$ and quits both auctions if this inequality fails. (This follows from Lemma 1 and Eqs. (1) and (2); with $n_\beta = 0$, $v^A_{\alpha\beta}(t^i_{\alpha\beta}, p_B, n_\beta) > p_A$ is equivalent to $t^i_{\alpha\beta} - p_B > p_A$, which is implied by $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) > p_B$, as $v^B_{\alpha\beta}(t^i_{\alpha\beta}, p_A, n_\alpha) \leq t^i_{\alpha\beta} - p_A$.) Thus, bidder $(\alpha\beta, i)$ keeps bidding for items A and B until the prices (p_A, p_B) reaches $(y_\alpha(t^i_{\alpha\beta}), y_\beta(t^i_{\alpha\beta}))$ (Figure 1). Either the highest $\alpha\beta$-bidder quits before the last α-bidder ($t^{(1)}_\beta \geq y_\alpha(t^{(1)}_{\alpha\beta})$) or else. The two terms in the square bracket of (5) correspond to these alternatives. In the first alternative, the global bidder wins item B when the highest $\alpha\beta$-bidder quits at $p_B = y_\beta(t^{(1)}_{\alpha\beta})$ (we assume that the global bidder has not quit, due to the aforementioned indicator function) and the global bidder bids for A until its price reaches $t^i_{\alpha\beta\gamma}$. In the second alternative, after all local bidders have quit, the highest $\alpha\beta$-bidder keeps bidding until $p_A + p_B$ reaches his type $t^{(1)}_{\alpha\beta}$. Thus, the global bidder's ex post profit from buying C now is equal to the right-hand side of (5), as claimed.

c. $n_\beta > 0 = n_\alpha$. This is analogous to the previous case (b), with (α, A) replaced by (β, B).

d. $n_\alpha n_\beta > 0$. I claim that the global bidder's ex post profit from buying C is equal to $w\left(t^i_{\alpha\beta\gamma}, t^{(1)}_{\alpha\beta}, t^{(1)}_\alpha\right) - p_C$. This claim implies that (11) follows from (10). To justify this claim, we examine Eq. (6), which defines w. With $n_\alpha n_\beta > 0$, a regional bidder $(\alpha\beta, i)$ keeps bidding for A and B until (p_A, p_B) reaches $(z_\alpha(t^{(1)}_{\alpha\beta}), z_\beta(t^{(1)}_{\alpha\beta}))$ (Lemma 1 and Eq. (3)). There are four alternatives:

   i. $(z_\alpha(t^{(1)}_{\alpha\beta}), z_\beta(t^{(1)}_{\alpha\beta})) \leqq (t^{(1)}_\alpha, t^{(1)}_\beta)$. Then eventually the global bidder becomes the only regional bidder competing only against the local bidders. Hence his ex post profit from buying C is equal to $u_C(t^i_{\alpha\beta\gamma}, t^{(1)}_\alpha, t^{(1)}_\beta) - p_C$, corresponding to the first term on the right-hand side of (6).

   ii. $z_\alpha(t^{(1)}_{\alpha\beta}) \le t^{(1)}_\alpha$ and $z_\beta(t^{(1)}_{\alpha\beta} > t^{(1)}_\beta$. Then eventually all β-bidders have quit while some α- and $\alpha\beta$-bidders remain active. This is the same as the previous case (b) ($n_\alpha > 0 = n_\beta$), where the ex post profit corresponds to the second term on the right-hand side of (6).

   iii. $z_\alpha(t^{(1)}_{\alpha\beta}) > t^{(1)}_\alpha$ and $z_\beta(t^{(1)}_{\alpha\beta} \le t^{(1)}_\beta$. This is analogous to the previous case (ii) and corresponds to the third term of (6).

   iv. $(z_\alpha(t^{(1)}_{\alpha\beta}), z_\beta(t^{(1)}_{\alpha\beta})) > (t^{(1)}_\alpha, t^{(1)}_\beta)$. Then eventually all local bidders quit while some $\alpha\beta$-bidders remain active. This is the same as the previous case (a) ($n_\alpha = n_\beta = 0$) and corresponds to the last term of (6).

Thus, all branches of Eq. (11) are proved, as desired. ■

Let us summarize the above lemmas and remarks into—

Proposition 1 *In the simultaneous ascending bid auctions such that resale is prohibited, an equilibrium is that local bidders all bid for their desired items up to their realized types and non-local bidders use the strategies described in Lemmas 1 and 2, and this is the only equilibrium where local bidders do not play weakly dominated strategies. The equilibrium allocation across bidders is inefficient with a a positive probability.*

The inefficiency of this equilibrium follows from the fact that the equilibrium maximum bids of non-local bidders are less than their types while local bidders bid up to their types.

This equilibrium is *straightforward* in the sense that the calculation of the equilibrium does not involve any fixed point argument. Based on local bidders' straightforward strategies, we calculate regional bidders' strategies. Based on local and regional bidders' strategies, which are independent of global bidders' strategies, we calculate global bidders' strategies. The above equilibrium analysis can be extended to a general environment with many items and many kinds of bidders that satisfies the following *nested* property: if S_i is the set of items desired by bidder i and S_j is the set of items desired by bidder j, then either $S_i \cap S_j = \emptyset$ or $S_i \subseteq S_j$ or $S_j \subseteq S_i$. With this nested property, a bidder knows that all his rivals whose desired packages contain his will quit once he has won a desired item. Hence his exposure problem comes only from rivals whose desired packages are proper subsets of his, so he needs only to guess these smaller bidders' equilibrium strategies. Thus, the equilibrium can be calculated via an induction starting from local bidders.

§3.2. The threshold problem of package auction

Let us analyze the package auction of items A and B that runs simultaneously with an English auction of item C, with resale prohibited for all auctions. Obviously it is weakly dominant for a regional bidder to bid for package $\{\mathrm{A}, \mathrm{B}\}$ until its price reaches its true value. A global bidder's equilibrium strategy can be calculated in a similar manner of §3.1, because to him the entire mechanism is simply simultaneous auctions of "item" $\{\mathrm{A}, \mathrm{B}\}$ and item C. The complexity comes from the threshold problem faced by local bidders, which creates an incentive for them to shade bids and form rational expectations on each other's bid shading strategies. For instance, suppose that remaining active are one α-bidder and several β- and non-local bidders. If the α-bidder quits at the current price p_A for $\{\mathrm{A}\}$ and if the non-local bidders are eventually defeated, then he wins item A only at his dropout price p_A even though the last non-local bidder's dropout price p_{AB} may be very high, since the winning β-bidder pays the difference $p_{AB} - p_A$. Hence the α-bidder may quit before p_A reaches his value, and the other bidders take that into account.

To focus on the threshold problem, I assume within this subsection that there is no global bidder. Hence the only issue is how local bidders coordinate their competition against regional bidders despite asymmetric information. The next observation is trivial.

Remark 4 *It is weakly dominant for a local bidder to bid straightforwardly (i.e., bid for his desired item up to his value) if at least one other local bidder of his kind remains active (bidding for the same item that he is bidding) or if all local bidders of the other kind (who desire the other item) have quit. It is weakly dominant for a regional bidder to bid straightforwardly for package* $\{A, B\}$.

To ensure existence of equilibrium, I assume within this subsection that every bidder has only finitely many possible types and prices rise in discrete increments. Specifically, assume that there is some small $\epsilon > 0$ such that every bidder's realized type is a multiple of ϵ, i.e., the realized type of a bidder of kind k ($k \in \{\alpha, \beta, \alpha\beta\}$) is equal to $s_k\epsilon$ for some integer $s_k = 0, 1, 2, \ldots, T_k$ such that $\epsilon T_k = \bar{t}_k$. Let $\pi_k(s_k)$ denote the probability with which a k-bidder's type is equal to $s_k\epsilon$. Assume that $\pi_k(s) = f_k(s\epsilon)\epsilon + O(\epsilon^2)$ for every $s = 0, 1, \ldots, T_k$. Call ϵ the *mesh* of the game. Amend the rules of the package auction (§2.2) as follows.

1. At each instant (or node) x, the package prices (p_A, p_B, p_{AB}) are given. A bidder who remains active up to x either quits immediately or stays. Then the next instant x' is reached with updated prices (p'_A, p'_B, p'_{AB}). If all α-local bidders have quit before x', $p'_A := p_A$; else $p'_A := p_A + \epsilon$. The case of p'_B is similar. The case of p'_{AB} is similar except that its increment is equal to $p'_A - p_A + p'_B - p_B$. (Hence it may be 2ϵ or ϵ.) If a bidder wins a package S at instant x' (because some one quits over the interval (x, x')), the winner's payment is equal to p'_S.
2. When the highest $\alpha\beta$-bidder ties with the highest local bidder (i.e., they quit in the same interval (x, x')), the $\alpha\beta$-bidder wins. (This merely simplifies calculations.)

Remark 5 *Given a realized type* $\epsilon s_{\alpha\beta}$ *of an* $\alpha\beta$*-bidder, let* $x(s_{\alpha\beta})$ *be the first node such that the price* p'_{AB} *at the next node is greater than* $\epsilon s_{\alpha\beta}$ *in the event that all other currently active bidders stay. An undominated strategy for this* $\alpha\beta$*-bidder is to keep bidding for package* $\{A, B\}$ *until node* $x(s_{\alpha\beta})$ *is reached and quit immediately after this node.*

Proof The only nontrivial case is that $s_{\alpha\beta} = 2m + 1$ for some integer m. At the node x where $p_{AB} = 2m\epsilon$ and both kinds of local bidders have active bidders up to x, the remark says it is undominated for the $\alpha\beta$-bidder to quit immediately after x. If at least one local bidder stays up to the next node, then the $\alpha\beta$-bidder's payment conditional on winning is at least as large as his value $(2m+1)\epsilon$, hence staying is unprofitable. If both local bidders quit immediately after node x, the increment of p_{AB} is zero, and he wins the package at this price whether he quits or not (the tie-breaking rule). ■

Let us consider only the basic case of having one bidder for each kind ($n_{\alpha\beta} = n_\alpha = n_\beta = 1$), as it can be easily extended to the general case by Remark 4, which obviously remains true in the discrete case. Given the $\alpha\beta$-bidder's undominated strategy described by Remark 5, the auction game is reduced to the following finite extensive form game between the α- and β-bidders.

1. A strategic node is indexed by an integer $x = 0, 1, \ldots, \min\{T_\alpha, T_\beta, T_{\alpha\beta}\}$, which indicates the current prices $p_A = p_B = p_{AB}/2 = x\epsilon$ and the event that all three bidders have been active up to x. Between x and the next node, an active bidder, via independent decision, either quits or stays. (If $x = T_k$, the k-bidder can only quit.) If no one quits, the next node is reached and this step is repeated with the update $x := x + 1$.

2. If both local bidders quit, $\{\mathrm{A}, \mathrm{B}\}$ goes to the $\alpha\beta$-bidder and the local bidders each get zero payoff. (This includes the case where the $\alpha\beta$-bidder quits simultaneously, by the tie-breaking rule). The game ends.

3. If the $\alpha\beta$-bidder quits and at least one local bidder stays, item A goes to the α-bidder and B to the β-bidder. The local bidder who quits pays $x\epsilon$; the local bidder who stays pays $(x+1)\epsilon$. The game ends.

4. If the $\alpha\beta$- and α-bidders stay and the β-bidder quits, the local bidders win if and only if $x + s_\alpha > s_{\alpha\beta}$ (Remark 4 and the tie-breaking rule), hence the α-bidder's payoff is equal to $\epsilon[s_\alpha - (s_{\alpha\beta} - x)]^+$, and the β-bidder's payoff is equal to $\epsilon(s_\beta - x)$ if $s_\alpha > s_{\alpha\beta} - x$ and is equal to zero if $s_\alpha \leq s_{\alpha\beta} - x$. The game ends.

5. If the $\alpha\beta$- and β-bidders stay and the α-bidder quits, the payoff vector is the same as that in step 4 with the roles of α and β switched. The game ends.

Since sequential equilibrium exists for finite extensive form games, the next remark follows.

Remark 6 *In the discrete package auction in the basic case ($n_\alpha = n_\beta = n_{\alpha\beta} = 1$), there exists an equilibrium where the regional bidder always bids straightforwardly, the local bidders play a sequential equilibrium of the above extensive form game until one of them quit, after which the remaining local bidder bids straightforwardly.*

To analyze these equilibria, pick any one of them. For each local bidder $k = \alpha, \beta$, let $\mathbf{z}_k$ stand for the first node immediately after which bidder k quits conditional on the event that no bidder has quit up to that node. Given the strategy profile of k, $\mathbf{z}_k$ is a well-defined random variable for others. Given k's strategy profile and for any strategic node x, let $\sigma_k(x)$ be the probability with which bidder k quits immediately after this node conditional on the event that no bidder has quit up to that node.

Consider any strategic node x (with all three bidders still active and currently $p_A = p_B = p_{AB}/2 = x\epsilon$). Suppose temporarily that the α-bidder is choosing between quitting immediately after the current node x, shorthanded as "quitting now," versus "quitting next," staying up to the next node x' and quitting immediately after x' if the extensive form game does not end by then. We shall calculate the α-bidder's expected net gain $\delta_\alpha(x, s_\alpha \mid \sigma_\beta)$ from "quitting next" instead of "quitting now," given his type $s_\alpha\epsilon$. During the interval (x, x'), there are only three alternatives:

1. Both the $\alpha\beta$- and β-bidders stay. Quitting now gives the α-bidder an expected payoff $\epsilon(s_\alpha - x)\text{Prob}(x + s_\beta > s_{\alpha\beta})$, while quitting next gives $\epsilon(s_\alpha - x - 1)\text{Prob}(x + 1 + s_\beta > s_{\alpha\beta})$. Hence the expected net gain from quitting next instead of now is equal to

$$\varphi_\alpha(x, s_\alpha \mid \sigma_\beta) := \epsilon \mathrm{E}\left[\begin{array}{c} (s_\alpha - x)1_{x+\mathbf{s}_\beta \le \mathbf{s}_{\alpha\beta} < x+1+\mathbf{s}_\beta}(\mathbf{s}_\beta, \mathbf{s}_{\alpha\beta}) \\ -1_{x+1+\mathbf{s}_\beta > \mathbf{s}_{\alpha\beta}}(\mathbf{s}_\beta, \mathbf{s}_{\alpha\beta}) \end{array} \middle| \begin{array}{c} \mathbf{s}_{\alpha\beta} \ge 2x; \\ \mathbf{z}_\beta \ge x \end{array} \right]. \tag{12}$$

2. The β-bidder quits. For the α-bidder, quitting now yields zero payoff. His payoff from "quitting next" is equal to $\epsilon[s_\alpha - (s_{\alpha\beta} - x)]^+$ if the $\alpha\beta$-bidder stays and is equal to $\epsilon(s_\alpha - x - 1)$ if the $\alpha\beta$-bidder quits. Hence the expected net gain from quitting next instead of now is equal to

$$\psi_\alpha(x, s_\alpha \mid \sigma_\beta) := \frac{\sigma_\beta(x)}{\sum_{x' \ge x} \sigma_\beta(x')} \epsilon \mathrm{E}\left[\begin{array}{c} [s_\alpha - (\mathbf{s}_{\alpha\beta} - x)]^+ 1_{\mathbf{s}_{\alpha\beta} \ge 2(x+1)}(\mathbf{s}_{\alpha\beta}) \\ +(s_\alpha - x - 1)1_{2x \le \mathbf{s}_{\alpha\beta} < 2(x+1)}(\mathbf{s}_{\alpha\beta}) \end{array} \middle| \mathbf{s}_{\alpha\beta} \ge 2x \right]. \tag{13}$$

3. The $\alpha\beta$-bidder quits and the β-bidder stays. This event is negligible because its probability is $O(\epsilon)$ and bidder α's ex post payoffs from "next" versus "now" differ by only ϵ.

Thus, the α-bidder's expected net gain from the intertemporal substitution is equal to

$$\delta_\alpha(x, s_\alpha \mid \sigma_\beta) = \varphi_\alpha(x, s_\alpha \mid \sigma_\beta) + \psi_\alpha(x, s_\alpha \mid \sigma_\beta) + O(\epsilon^2). \tag{14}$$

Eq. (14) reveals a local bidder's trade-off. On one hand, the *free rider effect* $\varphi_\alpha(x, s_\alpha \mid \sigma_\beta)$ makes the bidder want to quit sooner thereby shifting the burden of outbidding regional bidders to the other local bidder. On the other hand, the *pivotal effect* $\psi_\alpha(x, s_\alpha \mid \sigma_\beta)$ makes him want to quit later: if the other local bidder is about to quit, the α-bidder cannot acquire the desired item unless he stays slightly longer.

To characterize a local bidder's best reply, denote

$$h_\alpha(x \mid \sigma_\beta) := \mathrm{E}\left[2x \le \mathbf{s}_{\alpha\beta} < 2x+2 \mid 2x \le \mathbf{s}_{\alpha\beta}\right] + \mathrm{E}\left[\mathbf{z}_\beta = x \mid \mathbf{z}_\beta \ge x\right].$$

It can be easily shown that the Bellman equation for a type-ϵs_α α-bidder is

$$V_\alpha(x, s_\alpha \mid \sigma_\beta) = \max\left\{0, \delta_\alpha(x, s_\alpha \mid \sigma_\beta) + (1 - h_\alpha(x, s_\alpha \mid \sigma_\beta))V_\alpha(x+1, s_\alpha \mid \sigma_\beta)\right\}. \tag{15}$$

The value function V_α assigns to every strategic node the bidder's maximum expected payoff from having stayed up to this node minus his expected payoff from quitting immediately after this node. If $V_\alpha(x, s_\alpha \mid \sigma_\beta) > 0$, the bidder does not quit immediately after x. Obviously,

$$V_\alpha(s_\alpha, s_\alpha \mid \sigma_\beta) = 0. \tag{16}$$

Lemma 3 *Given the β-bidder's strategy σ_β and the α-bidder's type ϵs_α, there exist nodes $x_*(s_\alpha), x^*(s_\alpha) = 0, 1, \ldots, s_\alpha$ such that bidder α stays bidding at least up to node $x_*(s_\alpha)$ (though may quit immediately after this node) and quits no later than node $x^*(s_\alpha)$ is reached (unless β has quit before α), where*

$$\begin{aligned}
H_x^x &:= 1; \\
H_x^{x+m} &:= \prod_{j=1}^{m}(1 - h_\alpha(x+j \mid \sigma_\beta)), \ \forall m = 1, 2, \ldots; \\
x_*(s_\alpha) &:= \max\{0\} \cup \left\{ x = 1, \ldots, s_\alpha : \forall z = 0, \ldots, x-1, \sum_{j=z}^{x-1} H_z^j \delta_\alpha(j, s_\alpha \mid \sigma_\beta) > 0 \right\}; \quad (17) \\
x^*(s_\alpha) &:= \max\{0\} \cup \left\{ x = 1, \ldots, s_\alpha : \forall z = 0, \ldots, x-1, \sum_{j=z}^{x-1} H_z^j \delta_\alpha(j, s_\alpha \mid \sigma_\beta) \ge 0 \right\}. \quad (18)
\end{aligned}$$

Furthermore, if nodes $x_*(s_\alpha)-1$ and $x^*(s_\alpha)-1$ exist,

$$\delta_\alpha\left(x_*(s_\alpha)-1, s_\alpha \mid \sigma_\beta\right) \;>\; 0 \;\geq\; \delta_\alpha(x_*(s_\alpha), s_\alpha \mid \sigma_\beta); \tag{19}$$

$$\delta_\alpha(x^*(s_\alpha)-1, s_\alpha \mid \sigma_\beta) \;\geq\; 0 \;>\; \delta_\alpha(x^*(s_\alpha), s_\alpha \mid \sigma_\beta). \tag{20}$$

Proof When the α-bidder reaches a strategic node, we call this node *dropout point* if quitting immediately is his only best action, and call the node *semi-dropout point* if quitting immediately is one of his best actions. Identify the dropout points and semi-dropout points by backward induction: First, the ending node s_α, with ϵs_α being the bidder's type, is both a dropout and semi-dropout point. Second, if $\delta_\alpha(x, s_\alpha \mid \sigma_\beta) \leq 0$ for all $x = m, \ldots, m+j$ for some semi-dropout point $m+j+1$, then m is a semi-dropout point; if these inequalities are all strict, then m is a dropout point. Third, if $\sum_{j=z}^{x} H_z^j \delta_\alpha(j, s_\alpha \mid \sigma_\beta) \leq 0$ for some semi-dropout point $x+1$, then z is a semi-dropout point; if this inequality is strict, then x is a dropout point. Continuing this process, we exhaust the sets of dropout and semi-dropout points. The minimum in the set of semi-dropout points is $x_*(s_\alpha)$, and the minimum in the set of dropout points is $x^*(s_\alpha)$. Then (17)–(20) follow. ■

Lemma 4 *A local bidder's equilibrium dropout price is weakly increasing in his type in the following sense: for any sufficiently small ϵ, given local bidder β's strategy σ_β, local bidder α's lowest dropout price $\epsilon x_*(s_\alpha)$ and highest dropout price $\epsilon x^*(s_\alpha)$ (defined in (17) and (18)) are weakly increasing in α's type ϵs_α; this is also true when α and β switch roles.*

Proof Note from (12) and (13) that $\varphi_\alpha(x, s_\alpha \mid \sigma_\beta)$ and $\psi_\alpha(x, s_\alpha \mid \sigma_\beta)$ are each strictly increasing in s_α, hence by (14) $\delta_\alpha(x, s_\alpha \mid \sigma_\beta)$ is also strictly increasing in s_α for all sufficiently small ϵ. The conclusion then follows from (17) and (18). ■

When the price of an item is only one node away from reaching its local bidder's value ($x = s_\alpha - 1$ for item A), by Lemma 3, the local bidder always quit immediately if he has not quit earlier. That simply follows from the fact that $\varphi_\alpha(s_\alpha - 1, s_\alpha \mid \sigma_\beta) < 0$ and $\psi_\alpha(s_\alpha - 1, s_\alpha \mid \sigma_\beta) = 0$. (In other words, the local bidder makes no profit if he stays for the next node, where the price reaches his value; whereas, quitting immediately would bring him an ϵ profit if the other local bidder outbids the regional bidder later.) Hence a local bidder shades his bid by at least ϵ.

Does the degree of bid shading shrinks to zero as $\epsilon \to 0$? The next lemma says No, hence bid shading as a consequence of package auction is not negligible. To see the intuition, notice that, coupled with bidder α's dropout price, the probability with which the other local bidder β outbids the regional bidder is positive, hence the free rider effect is bounded away from zero when p_A approaches bidder α's value from below. In contrast, as p_A tends to α's value, his profit conditional on acquiring item A shrinks and hence the pivotal effect shrinks, unless the distribution of bidder β's dropout points has an atom nearby, but such an atom would imply that bidder β's bid shading is bounded away from zero. Thus, as least one of the local bidders shades bids with a positive probability.

Lemma 5 *If f_k is positive and continuous on $[0, \bar{t}_k]$ for all $k \in \{\alpha, \beta, \alpha\beta\}$, then there exist $\theta > 0$ and $M > 0$ such that, no matter how small the mesh between nodes is, there is at least a local bidder and a probability M with which the bidder's dropout price is less than his type by at least θ.*

Proof The lemma is proved by showing that at least one of the following claims is true:

i. local bidder β shades bids as the lemma asserts;

ii. for any $t_\alpha < \bar{t}_\beta$, there exists $\theta > 0$ such that local bidder α quits immediately after any node where p_A reaches $t_\alpha - \theta$, no matter how many nodes there are in $[p_A, t_\alpha]$.

Let $\tilde{\sigma}_\beta$ be a weak*-limit of the family of β's equilibrium strategies σ_β (indexed by ϵ); let $\mathbf{Z}_\beta$ be the dropout price (a random variable) induced by $\tilde{\sigma}_\beta$. For every $t_\alpha \in [0, \bar{t}_\alpha]$ and for any $p \in [0, t_\alpha]$, let s^ϵ_α be the node in the ϵ-grid nearest to t_α, and let x^ϵ be the node in the ϵ-grid nearest to p, and likewise denote $s^\epsilon_{\alpha\beta}$. We have:

$$\lim_{\epsilon \to 0} \frac{\varphi_\alpha(x^\epsilon, s^\epsilon_\alpha \mid \sigma_\beta)}{\epsilon} = \mathrm{E}\left[(t_\alpha - p) f_{\alpha\beta}(p + \mathbf{t}_\beta) - (F_{\alpha\beta}(p + \mathbf{t}_\beta) - F_{\alpha\beta}(2p)) \mid \mathbf{Z}_\beta \geq p\right]; \qquad (21)$$

$$\lim_{\epsilon \to 0} \epsilon \mathrm{E}\left[\begin{array}{c} [s^\epsilon_\alpha - (\mathbf{s}^\epsilon_{\alpha\beta} - x^\epsilon)]^+ 1_{\mathbf{s}^\epsilon_{\alpha\beta} \geq 2(x^\epsilon+1)}(\mathbf{s}^\epsilon_{\alpha\beta}) \\ +(s^\epsilon_\alpha - x^\epsilon - 1) 1_{2x^\epsilon \leq \mathbf{s}^\epsilon_{\alpha\beta} < 2(x^\epsilon+1)}(\mathbf{s}^\epsilon_{\alpha\beta}) \end{array} \middle| \, \mathbf{s}^\epsilon_{\alpha\beta} \geq 2x^\epsilon \right] = \mathrm{E}\left[[t_\alpha - \mathbf{t}_{\alpha\beta} + p]^+ \mid \mathbf{t}_{\alpha\beta} \geq 2p\right] . (22)$$

Pick any $t_\alpha < \bar{t}_\beta$. Notice that for all $p \in [0, t_\alpha]$,

$$\begin{aligned} \mathrm{E}\left[F_{\alpha\beta}(p+\mathbf{t}_\beta) - F_{\alpha\beta}(2p) \mid \mathbf{Z}_\beta \geq p\right] &\geq \mathrm{E}\left[F_{\alpha\beta}(p+\mathbf{t}_\beta) - F_{\alpha\beta}(2p)\right] \\ &= \mathrm{E}\left[f_{\alpha\beta}(2p+\xi(\mathbf{t}_\beta))(\mathbf{t}_\beta - p)\right] \end{aligned}$$

for some $\xi(\mathbf{t}_\beta) \in (2p, p+\mathbf{t}_\beta)$. Since $f_{\alpha\beta} > 0$ on its support and $p \leq t_\alpha < \bar{t}_\beta$, we have

$$\mathrm{E}\left[F_{\alpha\beta}(p+\mathbf{t}_\beta) - F_{\alpha\beta}(2p) \mid \mathbf{Z}_\beta \geq p\right] \geq L$$

for some positive constant L, for all $p \in [0, t_\alpha]$. As the continuous $f_{\alpha\beta}$ is uniformly bounded on its compact support, there exists $\theta_1 > 0$ for which if $t_\alpha - \theta_1 < p \leq t_\alpha$ then

$$(t_\alpha - p)\mathrm{E}\left[f_{\alpha\beta}(p+\mathbf{t}_\beta) \mid \mathbf{Z}_\beta \geq p\right] < L/3,$$

hence Eq. (21) implies that for all sufficiently small $\epsilon > 0$,

$$\frac{\varphi_\alpha(x^\epsilon, s_\alpha^\epsilon \mid \sigma_\beta)}{\epsilon} < -\frac{2}{3}L. \tag{23}$$

Suppose the limit distribution $\tilde{\sigma}_\beta$ of β's dropout prices has no atom in a neighborhood of t_α. Then $\frac{\sigma_\beta(x^\epsilon)/\epsilon}{\sum_{x' \geq x^\epsilon} \sigma_\beta(x')}$ is bounded by some constant when $\epsilon \to 0$. Since

$$\mathrm{E}\left[[t_\alpha - \mathbf{t}_{\alpha\beta} + p]^+ \mid \mathbf{t}_{\alpha\beta} \geq 2p\right] \leq t_\alpha - p,$$

there exists $\theta_2 > 0$ such that $t_\alpha - \theta_2 \leq p \leq t_\alpha$ implies

$$\lim_{\epsilon \to 0} \frac{\psi_\alpha(x^\epsilon, s_\alpha^\epsilon \mid \sigma_\beta)}{\epsilon} \mathrm{E}\left[[t_\alpha - \mathbf{t}_{\alpha\beta} + p]^+ \mid \mathbf{t}_{\alpha\beta} \geq 2p\right] < L/3$$

and hence, by (22), for all sufficiently small $\epsilon > 0$,

$$\frac{\psi_\alpha(x^\epsilon, s_\alpha^\epsilon \mid \sigma_\beta)}{\epsilon} < L/3. \tag{24}$$

Let $\theta := \min\{\theta_1, \theta_2\}$. If $t_\alpha - \theta \leq p \leq t_\alpha$, then for all sufficiently small ϵ,

$$\frac{\delta_\alpha(x^\epsilon, s_\alpha^\epsilon \mid \sigma_\beta)}{\epsilon} < -L/3$$

by (23) and (24), and so bidder α quits immediately after the price of A has reached p (Lemma 3). Hence the above-listed alternative (ii) is true.

Finally, consider the other case, where the limit distribution $\tilde{\sigma}_\beta$ has an atom arbitrarily near to t_α. Let $2M > 0$ be the mass of the atom. Denote G_β for the cumulative distribution function of bidder β's dropout price. Then for any arbitrarily small $\eta > 0$

$$G_\beta(t_\alpha) - G_\beta(t_\alpha - \eta) > 2M.$$

Let F^{ϵ}_{β} be the cumulative distribution of bidder β's type when ϵ is the mesh of the discrete package auction game. By individual rationality, $G_{\beta}(t_{\alpha} - \eta) \geq F^{\epsilon}_{\beta}(t_{\alpha} - \eta)$. Hence

$$G_{\beta}(t_{\alpha}) > F^{\epsilon}_{\beta}(t_{\alpha} - \eta) + 2M$$

for any $\eta > 0$. As $\epsilon \to 0$, F^{ϵ}_{β} converges to F_{β}, which is atomless by assumption, hence the atoms of F^{ϵ}_{β} become arbitrarily small. Thus, for any $\zeta > 0$ there exists $\eta > 0$ for which $F^{\epsilon}_{\beta}(t_{\alpha} - \eta) > F^{\epsilon}_{\beta}(t_{\alpha}) - \zeta$ for all sufficiently small ϵ. Hence

$$G_{\beta}(t_{\alpha}) \geq F^{\epsilon}_{\beta}(t_{\alpha}) + 2M.$$

Consequently, there is a probability-$2M$ set of types such that they are greater than t_{α} and bidder β with such types quits when the price for B reaches t_{α}. For all sufficiently small ϵ, the atoms of F^{ϵ}_{β} are arbitrarily small, yet the mass $2M$ of these bid-shading types stays constant. Hence some of these types, of mass M, must be greater than t_{α} by a distance $\theta' > 0$ for all sufficiently small ϵ. Hence alternative (i) is true. This proves the lemma. ■

The above analysis of the basic case is now extended to allow multiple players for each kind of local bidders. When another local bidder bidding for the same item is also active, a local bidder is effectively playing an English auction: he wins nothing if quitting before the price reaches his value, because the higher local bidder who is bidding for the same item wins it if regional bidders are defeated.

Proposition 2 *In the discrete package auction such that resale is prohibited, if global bidders are absent and if f_k is positive and continuous on $[0, \bar{t}_k]$ for all $k \in \{\alpha, \beta, \alpha\beta\}$, there exists an equilibrium with these properties:*

1. *regional bidders bid straightforwardly;*
2. *a local bidder bids straightforwardly if at least one other local bidder of his kind remains active or if all local bidders of the other kind have dropped out;*
3. *if he is the only remaining α-bidder and if more than one β-bidder remains active, an α-bidder with type t^{i}_{α} quits when the price for $\{A\}$ reaches $t^{i}_{\alpha} - \theta$, where $\theta > 0$ may depend on t_{α} but is independent of the mesh ϵ of the mesh of strategic nodes; the case with a single remaining β-bidder and multiple α-bidders is analogous;*

4. *if there are exactly one α- and one β-bidders left, at least one of them shades bids with a positive probability such that the amount of bid-shading and the probability are both bounded away from zero no matter how small the mesh between nodes is;*

5. *a local bidder's lowest and highest dropout prices are weakly increasing in his type;*

6. *the allocation across bidders is inefficient with a positive probability.*

Proof The only part not yet proved previously is claim 3. That claim can be proved by the proof for alternative (ii) in Lemma 5. That proof is applicable because the distribution of the β-bidders' dropout prices is atomless when there are multiple remaining β-bidders: the β-bidders are straightforward and hence their dropout prices are the same as their realized types, which are assumed to be atomless. ■

By the proof of Lemma 5, at the limit as $\epsilon \to 0$, one local bidder drops out at zero price with a positive probability bounded away zero while the other local bidder has zero probability of quitting at zero price. Thus, when the mesh of the discrete auction is sufficiently small, there is no symmetric equilibrium where the local bidders use the same strategy.

Corollary 1 *In the above package auction with the basic case, if f_k is positive and continuous on $[0, \bar{t}_k]$ for all $k \in \{\alpha, \beta, \alpha\beta\}$, then for all sufficiently small mesh ϵ between nodes and at any equilibrium, bidder β's probability of quitting when $p_B = 0$ is $O(\epsilon)$ if and only if bidder α's probability of quitting at $p_A = 0$ is a positive constant no matter how small ϵ is.*

Proof The "only if" part follows from the proof of Lemma 5. (Just use the fact in that proof that alternative (ii) is true if alternative (i) is not; then consider t_α nearby zero.) For the "if" part, suppose bidder α quits at zero price with a probability M bounded away from zero. Then bidder β with any positive type can do better than quitting at zero price by quitting at a price slightly above zero. In taking the latter action, bidder β increases his probability of winning by M and pays only a slightly higher price if he wins. ■

§4. Equilibria when resale is allowed

The previous section has shown that both auction mechanisms result in inefficient allocations with positive probabilities. Thus, by the same reasoning of Zheng [11, Prop. 1], resale occurs with positive probabilities if the option of resale is available to bidders. In the following I report my preliminary analysis on these mechanisms when resale is allowed.

Since a bidder's action in period one will be used to update information about him, it is obvious that a bidder who expects a positive probability of buying an item at resale has an incentive to conceal his true value by shading his bids in period one. One type of bid shading that facilitates tractability is not to bid in period one at all. This strategy is a best reply if the bidder expects some other bidder to bid for all items no matter how high the prices become. Although such bidding behavior also constitutes an equilibrium in the case where resale is prohibited, it is weakly dominated there. In contrast, when resale is allowed, such bidding behavior is not weakly dominated. That is because a high bidder can consistently believe that his resale revenue can cover his payments for the items, and the bidders who shy away in period one can consistently believe that entering a bid in period one can only result in being charged a higher price at resale. (This is similar to the point made by Garratt and Tröger [5] for a single-good model.)

§4.1. The case with two items

Let us start with the case of only two items, A and B, and three kinds of bidders, local bidders for A, local bidders for B, and regional bidders for $\{\mathrm{A}, \mathrm{B}\}$. As in Zheng [11], an *allocation* from a bidder i's standpoint is a mapping from the profile of types across players other than i to a lottery that assigns the items to players (including i). This mapping's image of a type-profile is called an *allocation outcome*. An allocation is said to be *Myerson* from bidder i's standpoint if it maximizes i's expected profit among all allocations from i's standpoint that are Bayesian incentive feasible given i's belief. Given a type-profile, the *virtual utility* of an allocation outcome from bidder i's standpoint is defined to be the ex post gain of trade generated by this outcome minus

$$\sum_{j \notin i} \frac{1 - F_j(t_j)}{f_j(t_j)},$$

where index j ranges through all players but i who are involved in the trade specified by the allocation outcome, and t_j is the realized type of such a player j. For instance, the virtual utility of giving item A to bidder (α, i) and item B to bidder (β, j), from bidder $(\alpha\beta, l)$'s standpoint, is equal to

$$t_\alpha^i + t_\beta^j - t_{\alpha\beta}^l - \frac{1 - F_\alpha(t_\alpha^i)}{f_\alpha(t_\alpha^i)} - \frac{1 - F_\beta(t_\beta^j)}{f_\beta(t_\beta^j)}.$$

The next lemma can be proved by the standard technique of optimal auctions extended to multiple-object environments by Levin [7]. The standard technique is applicable because each bidder's private information is assumed to be one-dimensional.

Lemma 6 *Suppose that for every $k \in \{\alpha, \beta, \alpha\beta\}$, the hazard rate $f_k/(1 - F_k)$ for bidders of kind k is weakly increasing. Then an allocation from bidder i's standpoint is Myerson if and only if, for almost every type-profile, the allocation outcome maximizes the virtual utility from i's standpoint among all allocation outcomes from i's standpoint.*

Proposition 3 *Suppose the hazard rate $f_k/(1 - F_k)$ for local bidders of kind k ($k = \alpha, \beta$) is weakly increasing. Then there is an equilibrium in the auction-resale game whether the period-one mechanism is simultaneous auctions or package auction; in this equilibrium, all regional bidders participate in the period-one auctions and all other bidders do not, the regional bidder with the highest type wins all items in period one and offers resale to local bidders according to the Myerson allocation from his standpoint. There is no further resale after the operation of this reseller's mechanism, and the final allocation is incentive efficient.*

Proof In period one, a regional bidder's maximum bid is equal to his realized type plus the maximum expected profit that he can obtain during the resale stage. If the period-one mechanism is simultaneous auctions, he keeps bidding for both items until $p_A + p_B$ reaches his maximum bid. If the mechanism is package auction, he keeps bidding for $\{\text{A}, \text{B}\}$ until p_{AB} reaches his maximum bid. Once all but one regional bidder have quit, the remaining regional bidder will continue bidding forever should there be remaining local bidders. Note that the maximum bid is strictly increasing in the regional bidder's type. (This can be proved by mimicking the envelope theorem argument in the proof of Proposition 2 in Zheng [11]—from the start of that proof down to its second displayed equation.) Hence the winner, say $(\alpha\beta, 1)$, is the one whose type is highest among all regional bidders, so he will not include the other regional bidders into potential buyers in his resale mechanism.

Since local bidders do not participate in the auctions in period one, the winner $(\alpha\beta, 1)$ maintains his prior belief about them. Hence he achieves his optimum if the Myerson allocation from his standpoint can be implemented via a perfect Bayesian equilibrium (PBE) starting from period two. Since the Myerson allocation is Bayesian incentive feasible, it is implementable via PBE if it leaves zero gain of trade among players other than $(\alpha\beta, 1)$.

To complete the proof, I show that the Myerson allocation leaves zero gain of trade among players other than $(\alpha\beta, 1)$. One can show that the Myerson allocation has only four possible allocation outcomes: (i) the reseller $(\alpha\beta, 1)$ keeps both items, (ii) he keeps item A and sells B to the highest-type β-bidder say $(\beta, 1)$, (iii) the reseller keeps B and sells item A to the highest-type α-bidder say $(\alpha, 1)$, or (iv) item A goes to $(\alpha, 1)$ and B goes to $(\beta, 1)$. (The allocation is unbiased because local bidders of the same kind have the same distribution and hence the same virtual utility function.) As a reseller is assumed to be able to commit to keeping his goods, outcome (i) does not lead to resale. In the other allocation outcomes, there is no gain of trade from further resale. Say the highest local bidder α gets item A. He cannot profit from selling A to other α-bidders, since they value A less, or to β-bidders, who do not value A, or to the $\alpha\beta$-bidders, whose value of A alone never exceeds α's value. The only case left is (iv), where the two winning local bidders could resell both items to a single regional bidder. But this trade leads to no surplus either, because the fact that the Myerson allocation gives the two items to the two local bidders implies that

$$t_\alpha^1 + t_\beta^1 > t_\alpha^1 + t_\beta^1 - \frac{1 - F_\alpha(t_\alpha^1)}{f_\alpha(t_\alpha^1)} - \frac{1 - F_\beta(t_\beta^1)}{f_\beta(t_\beta^1)} \geq t_{\alpha\beta}^1 \geq t_{\alpha\beta}^m$$

for all regional bidders $(\alpha\beta, m)$. ■

With final allocation being Myerson with respect to prior beliefs, the above equilibrium is incentive efficient (Myerson [9, Ch. 10]) relative to priors. This property, however, relies on the condition that the Myerson allocation with respect to priors is implementable. This condition may be violated in general. For instance, consider a different equilibrium where the winner who takes all in period one is an α-local bidder instead of a regional bidder.

Suppose this reseller has decided to keep item A to himself and is choosing between whether to sell item B to a β-bidder or to a regional bidder. According to the Myerson allocation, the reseller awards B to the regional bidder with a positive probability. This allocation outcome is biased, because the standalone value of B for the regional bidder can never be higher than its value for a β-bidder. Hence the Myerson allocation from α's standpoint cannot be implemented without further resale. Such resale creates additional incentive constraints, which may make it impossible to implement the Myerson allocation.

§4.2. The general case

Let us bring item C and the global bidders back to the picture. For simplicity, assume that every bidder's type space is a finite grid with mesh $\epsilon > 0$ and that payments can only be made in multiples of ϵ, as in §3.2. A *social outcome* consists of an assignment of items to players and an arrangement of payments. A *mechanism* from bidder i's standpoint is restricted to be a mapping that assigns a social outcome to every profile of types across players other than i. By this formulation, there are only finitely many possible types, social outcomes, and mechanisms. This, coupled with the assumption that there are only a finite exogenous number of periods, implies that the entire auction-resale game is a finite extensive form game such that a reseller chooses a mechanism and non-reseller players pick types to report. Hence a reseller's optimal resale mechanism is part of his strategy in a sequential equilibrium of the entire game. By existence of sequential equilibrium, we have—

Lemma 7 *For every $k \in \{\alpha, \beta, \alpha\beta, \alpha\beta\gamma, \gamma\}$ and given any prior, a bidder-turned reseller's problem of finding an expected-profit maximizing mechanism subject to the constraints of (sequential) equilibrium feasibility has an optimal solution.*

Proposition 4 *For any kind of bidders (global, regional, α-local, etc.) there is an equilibrium in the discrete auction-resale game such that the highest-type bidder of this kind wins all items in period one, while bidders of the other kinds bid zero, and the winner offers resale to bidders of the other kinds via an optimal mechanism. Furthermore, this equilibrium generates the same final outcome and period-one revenue whether the period-one mechanism is simultaneous auctions or package auction.*

Proof Using Lemma 7 and mimicking the first paragraph in the proof of Proposition 3, we deduce that the bidding behavior described above constitutes an equilibrium. The two period-one mechanisms make no difference due to this bidding behavior. ■

§5. Discussion

The equilibria when resale is prohibited (§3) are *interactive* in the sense that calculation of a bidder's equilibrium strategy requires complete knowledge of other bidders' strategies. In contrast, the equilibria when resale is allowed (§4) are *individualized* in the sense that all bidders but a reseller act as a type-reporting agent in the resale mechanism and the reseller picks the mechanism that solves a constrained optimization problem. Everyone is merely solving a decision problem, as in Walras equilibrium. The task of computation is delegated to the reseller.

We can partially remove the assumption of having only three items. The analysis on simultaneous auctions that bans resale can be extended to n items if the nested property (§3.1) is satisfied. The assumption of having unidimensional private information can be removed for the general case that allows resale. That assumption is needed only to obtain explicit properties of the optimal mechanisms, while Proposition 4 needs only their existence.

To counterbalance a reseller's monopoly power, a government with limited policy instruments might impose item-specific price caps during the resale stage. The effects of such price caps may be worth investigating.

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